Journal of Computational and Applied Mathematics 272 (2014) 221–238

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Biorthogonal polynomials and numerical quadrature formulas for some finite-range integrals with symmetric weight functions

Avram Sidi *

Computer Science Department, Technion - Israel Institute of Technology, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 28 October 2012
Received in revised form 2 March 2014

MSC:
41A55
41A60
65B05
65B10
65B15
65D30

Keywords:
Biorthogonal polynomials
Numerical integration
Symmetric weight functions
Acceleration of convergence
Levin transformation
Rational approximation

ABSTRACT

In this work, we derive a family of symmetric numerical quadrature formulas for finite-range integrals $I[f] = \int_{-1}^{1} w(x)f(x)\,dx$, where $w(x)$ is a symmetric weight function. In particular, we will treat the commonly occurring case of $w(x) = (1-x^2)^{\alpha}[\log(1-x^2)^{-1}]^{p}$, p being a nonnegative integer. These formulas are derived by applying a modification of the Levin $\mathcal{L}$ transformation to some suitable asymptotic expansion of the function $H(z) = \int_{-1}^{1} w(x)/(z-x)\,dx$ as $z \to \infty$, and they turn out to be interpolatory. The abscissas of these formulas have some rather interesting properties: (i) they are the same for all α, (ii) they are real and in $[-1, 1]$, and (iii) they are related to the zeros of some known polynomials that are biorthogonal to certain powers of $\log(1-x^2)^{-1}$. We provide tables and numerical examples that illustrate the effectiveness of our numerical quadrature formulas.

1. Introduction and background

In [1,2], the author introduced a novel approach by which one can derive interpolatory numerical quadrature formulas of high accuracy for integrals of the form

$$I[f] = \int_a^b w(x)f(x)\,dx, \tag{1.1}$$

where (a, b) is a finite or infinite interval and $w(x)$ is a nonnegative weight function all of whose moments exist. The quadrature formulas are of the form

$$I_n[f] = \sum_{i=1}^{n} w_{ni} f(x_{ni}), \tag{1.2}$$

* Tel.: +972 4 8294364.
E-mail address: asidi@cs.technion.ac.il.
URL: http://www.cs.technion.ac.il/~asidi.

http://dx.doi.org/10.1016/j.cam.2014.05.013
0377-0427/

x_{ni} and w_{ni} being, respectively, the abscissas and weights. In this approach, we assume for simplicity that $[a, b]$ is a finite interval and that $f(z)$ is an analytic function in an open domain D of the z-plane that contains $[a, b]$ in its interior. Then, we can express $f(x)$ via

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{z-x}\, dz, \tag{1.3}$$

where Γ is a closed contour whose interior contains $[a, b]$ and is traversed counterclockwise. Substituting (1.3) in (1.1), and changing the order of integration, we obtain

$$I[f] = \frac{1}{2\pi \mathrm{i}} \int_\Gamma H(z) f(z)\, dz, \tag{1.4}$$

where we have defined

$$H(z) = \int_a^b \frac{w(x)}{z-x}\, dx. \tag{1.5}$$

Clearly, $H(z)$ is analytic in the complex z-plane cut along the line segment $[a, b]$. Next, substituting (1.3) in (1.2), and changing the order of summation and integration, we obtain

$$I_n[f] = \frac{1}{2\pi \mathrm{i}} \int_\Gamma H_n(z) f(z)\, dz, \tag{1.6}$$

where

$$H_n(z) = \sum_{i=1}^{n} \frac{w_{ni}}{z - x_{ni}}. \tag{1.7}$$

It is easy to see that $H_n(z)$ is a rational function with degree of numerator at most $n-1$ and degree of denominator n. In addition, the abscissas x_{ni} are the poles of $H_n(z)$, while the weights w_{ni} are the corresponding residues. By (1.4) and (1.6), we have

$$I[f] - I_n[f] = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \big[H(z) - H_n(z)\big] f(z)\, dz. \tag{1.8}$$

Thus, we conclude that in order for $I_n[f]$ to be a good approximation to $I[f]$ for arbitrary analytic $f(z)$, it is necessary that $H_n(z)$ be a good approximation to $H(z)$ in the z-plane cut along $[a, b]$. This means that, in deriving good numerical quadrature formulas of the form given in (1.2), we should construct rational functions $H_n(z)$ that will approximate $H(z)$ well in the z-plane cut along $[a, b]$.

Now, sequences of rational approximations to $H(z)$ can be generated in different ways. A good way is by applying appropriate *convergence acceleration methods* (or *sequence transformations*) to the sequence of partial sums of the moment series associated with $w(x)$, which is nothing but the asymptotic expansion of $H(z)$ as $z \to \infty$ in negative powers of z, namely,

$$H(z) \sim \sum_{m=1}^{\infty} \frac{\hat{\mu}_m}{z^m} \quad \text{as } z \to \infty; \qquad \hat{\mu}_m = \int_a^b w(x) x^{m-1}\, dx, \quad m = 1, 2, \ldots. \tag{1.9}$$

If the transformation of Shanks [3] is used for this purpose, the rational functions $H_n(z)$ generated by it are simply the $[n-1/n]$ Padé approximants from (1.9), and the resulting $I_n[f]$ are the Gaussian quadrature formulas for $I[f]$. For Padé approximants, see, for example, Baker [4] and Baker and Graves-Morris [5]. For a brief review, see also Sidi [6, Chapter 17]. For an up-to-date treatment of the Shanks transformation, see [6, Chapter 16]. For Gaussian quadrature, see Davis and Rabinowitz [7], Stoer and Bulirsch [8], or Ralston and Rabinowitz [9], for example.

In [1], a suitably modified version of the $\mathcal{L}$ transformation of Levin [10] is applied to the sequence of partial sums $\{\sum_{i=1}^{m} \hat{\mu}_i z^{-i}\}_{m=1}^{\infty}$ of the moment series in (1.9) with $(a, b) = (0, 1)$ and $w(x) = (1-x)^\alpha x^\beta (\log x^{-1})^\nu$, and good numerical quadrature formulas for $I[f]$ are obtained. These formulas also turn out to be interpolatory. One interesting and useful feature of these formulas is that their abscissas are *independent of* β; they can also be made *independent of* α and ν provided $\alpha + \nu$ is a small nonnegative integer. In a recent paper by Lubinsky and Sidi [11], it is shown, for example, that the weights w_{ni} associated with these quadrature formulas are *positive* when $\alpha = 0$ and $\beta, \nu > -1$. This is a very important property in that it implies that $\lim_{n\to\infty} I_n[f] = I[f]$ for every $f \in C[a, b]$ since $I_n[f]$ is also interpolatory. See Krylov [12, p. 264, Theorem 8], for example.

The approach of [1] is used in [2] for obtaining numerical quadrature formulas for the infinite-range integrals $I^{(i)}[f] = \int_0^\infty w_i(x) f(x)\, dx$, with $w_1(x) = x^\alpha e^{-x}$ and $w_2(x) = x^\alpha E_p(x)$, where $E_p(x) = \int_1^\infty e^{-xt} t^{-p}\, dt$ is the Exponential Integral and p is *arbitrary*. Yet in another recent paper by Sidi and Lubinsky [13], we treat the same integrals as in [2], only this time we use a suitably modified version of the $\mathcal{S}$ transformation of Sidi instead of the $\mathcal{L}$ transformation. (For the $\mathcal{L}$ and $\mathcal{S}$ transformations, see Sidi [6, Chapter 19].) The quadrature formulas developed in [2,13], just as those developed in [1], also turn out to be interpolatory. Their abscissas have properties similar to those in [1]: the abscissas for $I^{(2)}[f]$ are *independent of* p and they are the *same* as those of $I^{(1)}[f]$.

Before proceeding further, we note that the computation of the abscissas x_{ni} is the most substantial part of the determination of the quadrature formula $I_n[f]$ as the x_{ni} are the zeros of a polynomial of degree n an hence their determination is costly for large n. Once the abscissas have been determined, the weights can be computed as the residues of $H_n(z)$ at a negligible cost. This is why the fact that the abscissas of the quadrature formulas above are independent of β and of p, for example, is so useful.

Another interesting feature of the quadrature formulas derived in [1,2,13], and mentioned above, is that their abscissas are the zeros of some polynomials that are *biorthogonal* to (i) some powers of $\log x^{-1}$ (in [1]), as shown in Sidi and Lubinsky [14], and (ii) some exponential functions (in [2,13]), as shown in Sidi [15] and in Sidi and Lubinsky [13].[1] These polynomials have interesting asymptotic properties and zero distributions, which are studied in Lubinsky and Sidi [16,17].

In [1, Section 7], again a suitably modified version of the $\mathcal{L}$ transformation of Levin [10] is applied to the moment series in (1.9) with $(a, b) = (-1, 1)$ and $w(x) = (1-x^2)^\alpha$, and good numerical quadrature formulas for $I[f]$ are obtained. Just as the weight function, the quadrature formulas derived in this way too are symmetric, that is, if (ξ, w) is an abscissa-weight pair, then so is $(-\xi, w)$. Unfortunately, however, the abscissas of these quadrature formulas *vary* with α. The purpose of the present work is to derive symmetric quadrature formulas that do not suffer from this deficiency. To achieve this, we modify the approach of [1] substantially by considering an asymptotic expansion of $H(z)$ that is very different from that in (1.9).

In the next section, we develop this modified approach and apply it in the presence of symmetric weight functions of the form $w(x) = (1-x^2)^\alpha[\log(1-x^2)^{-1}]^p$, $p > -1/2$ (that are more general than $w(x) = (1-x^2)^\alpha$) and derive a sequence of rational approximations to $H(z)$. In Section 3, we use these rational approximations to derive our quadrature formulas. We study some of the properties of these formulas and show that their abscissas are real and lie in $[-1, 1]$, those in $(-1, 1)$ being the zeros of polynomials that are biorthogonal to some powers of $\log(1-x^2)^{-1}$. The abscissas ±1, if present, can have multiplicities ≥ 1. In Section 4, we show that these formulas are interpolatory. In Section 5, we derive integral representations for the weights of the quadrature formulas. In Section 6, we give some tables of abscissas and weights and also demonstrate the performance of the new formulas with some numerical examples.

As we will be applying the Levin $\mathcal{L}$ transformation later in this work, we provide a brief description of the essentials of it in the Appendix to this work. This description should also help the reader to better understand the motivation for the developments in the next section, hence we advise the reader to study it first.

Before closing, we would like to comment on the relevance of the weight functions treated in this paper. The weight function $w(x) = (1-x^2)^{\beta-1/2}$, $-1 < x < 1$, is associated with Gegenbauer polynomials (or ultraspherical polynomials) $C_n^{(\beta)}(x)$, which are of importance in the context of potential theory and harmonic analysis. They are also used in numerical work just as the Chebyshev and Legendre polynomials, for example. For these and other orthogonal polynomials, see Szegő [18] or Olver et al. [19], for example. The weight function $w(x) = (1-x^2)^\alpha[\log(1-x^2)^{-1}]^p$, with $p = 1, 2, \ldots,$ arises as in

$$\frac{d^p}{d\alpha^p}\int_{-1}^{1}(1-x^2)^\alpha f(x)\,dx = (-1)^p\int_{-1}^{1}(1-x^2)^\alpha[\log(1-x^2)^{-1}]^p f(x)\,dx.$$

Finally, we would like to state that when the abscissas and weights of Gaussian quadrature formulas for the integrals we deal with in this work are available, Gaussian formulas should be used as they provide "optimal" accuracies for functions $f(x)$ that are infinitely smooth on $[-1, 1]$. This, of course, requires the determination of the appropriate abscissas and weights for each α and p, which may prove to be cumbersome. As already mentioned, for the quadrature formulas we develop in this work, the abscissas, which are the most important quantities, are *independent of* α, and also *independent of* p when p is an integer. This means that one set of abscissas (with $p = 0$) is good for *all* α and integer p. As we will see later, the abscissas are obtained as the zeros of some simple polynomials, the determination of the weights being a trivial task once the abscissas are available. In addition, numerical experience shows that these quadrature formulas are capable of achieving very high accuracies, and this is sufficient to justify their practical use.

2. Rational approximations to $H(z)$

Throughout the remainder of this work, we will consider the integrals

$$I[f] = \int_{-1}^{1} w(x)f(x)\,dx, \tag{2.1}$$

where the weight function $w(x)$ is an even function of x, that is,

$$w(-x) = w(x), \quad -1 < x < 1. \tag{2.2}$$

[1] A polynomial $P(x)$ of degree n is said to be *biorthogonal* to a set of (not necessarily polynomial) functions $\{\phi_1(x), \ldots, \phi_n(x)\}$ in the inner product $(F, G) = \int_a^b w(x)F(x)G(x)dx$, if it satisfies $(P, \phi_j) = 0$, $1 \leq j \leq n$.

If $\phi_j(x) = x^{j-1}$, $1 \leq j \leq n$, then $P(x)$ is the nth *orthogonal* polynomial in the inner product $(F, G) = \int_a^b w(x)F(x)G(x)dx$. In this case, the quadrature method obtained is the Gaussian rule for $\int_a^b w(x)f(x)\,dx$ mentioned following (1.9).

In addition, we assume that all moments of $w(x)$ exist. We aim at obtaining numerical quadrature formulas for $I[f]$ that are *interpolatory and symmetric*, and hence of the form

$$I_n[f] = \sum_{i=1}^{n} w_{ni}\big[f(x_{ni}) + f(-x_{ni})\big].\text{[2]} \tag{2.3}$$

In particular, we will treat the commonly occurring case of

$$w(x) = (1-x^2)^{\alpha}\,[\log(1-x^2)^{-1}]^p, \quad \alpha > -1,\ p > -1/2.\text{[3]} \tag{2.4}$$

Let us first define

$$H(z) = \int_{-1}^{1} \frac{w(x)}{z-x}\,dx. \tag{2.5}$$

Clearly, $H(z)$ is analytic in the complex z-plane cut along the line segment $[-1, 1]$. On account of (2.2), we also have that

$$H(z) = \int_{-1}^{1} \frac{w(x)}{z+x}\,dx.$$

As a result,

$$H(z) = \frac{1}{2}\int_{-1}^{1} w(x)\left[\frac{1}{z-x} + \frac{1}{z+x}\right]dx,$$

from which, we have

$$H(z) = z\int_{-1}^{1} \frac{w(x)}{z^2-x^2}\,dx. \tag{2.6}$$

The next lemma provides an asymptotic expansion for $H(z)$ as $z \to \infty$ that is entirely different from that in (1.9).

Theorem 2.1. *Assume that $w(x)$ satisfies (2.2), and let*

$$\mu_m = \int_{-1}^{1} w(x)(1-x^2)^{m-1}\,dx, \quad m = 1, 2, \ldots. \tag{2.7}$$

Then $H(z)$ has the convergent expansion

$$H(z) = -z\sum_{m=1}^{\infty} \mu_m (1-z^2)^{-m} \quad \text{when } |1-z^2| > 1. \tag{2.8}$$

Thus, the right-hand side of (2.8) also represents $H(z)$ asymptotically as $z \to \infty$.

Proof. We begin by rewriting (2.6) in the form

$$H(z) = -z\int_{-1}^{1} \frac{w(x)}{(1-z^2)-(1-x^2)}\,dx = -\frac{z}{1-z^2}\int_{-1}^{1} \frac{w(x)}{1-\frac{1-x^2}{1-z^2}}\,dx. \tag{2.9}$$

Now, because $|1-x^2| \le 1$ when $x \in [-1, 1]$,

$$\frac{1}{1-\frac{1-x^2}{1-z^2}} = \sum_{m=1}^{\infty}\left(\frac{1-x^2}{1-z^2}\right)^{m-1} \quad \text{when } |1-z^2| > 1. \tag{2.10}$$

Substituting (2.10) in (2.9), integrating termwise, and invoking (2.7), we obtain (2.8). Clearly, the infinite series in (2.10) converges for all large z, and so does the right-hand side of (2.8). This completes the proof. ■

We next analyze the asymptotic behavior of μ_m as $m \to \infty$ when $w(x)$ is as in (2.4). We need this in order to decide whether the $\mathcal{L}$ transformation can be used to accelerate the convergence of the infinite series in (2.8).

[2] Note that if $w(x)$ is as in (2.2) and the abscissas of $I_n[f]$ are $\pm x_{ni}$ and $I_n[f]$ is interpolatory, then $I_n[f]$ is as in (2.3). We give an independent proof of this in Theorem 5.1.

[3] The condition $\alpha > -1$ is needed to make $w(x)$ integrable at $x = \pm 1$. The condition $p > -1/2$ is needed to make $w(x)$ integrable through $x = 0$ since $[\log(1-x^2)^{-1}]^p \sim |x|^{2p}$ as $|x| \to 0$.

Theorem 2.2. *Let $w(x)$ be exactly as in (2.4). Then μ_m has the asymptotic expansion*

$$\mu_m \sim \sum_{j=0}^{\infty} (-1)^j \frac{B_j^{(1/2)}(\alpha)}{j!} \frac{\Gamma(j+p+1/2)}{m^{j+p+1/2}} \quad \text{as } m \to \infty. \tag{2.11}$$

Here $B_s^{(\sigma)}(u)$ are generalized Bernoulli polynomials.[4]

Proof. We start by noting that, due to (2.2), (2.7) can be rewritten in the form

$$\mu_m = 2\int_0^1 w(x)(1-x^2)^{m-1}\,dx, \quad m = 1, 2, \ldots. \tag{2.12}$$

Making the change of variable of integration $1 - x^2 = e^{-t}$ in (2.12), we obtain

$$\mu_m = \int_0^{\infty} e^{-mt} g(t)\,dt, \qquad g(t) = \left(\frac{t}{e^t - 1}\right)^{1/2} e^{(1/2-\alpha)t}\, t^{p-1/2}. \tag{2.13}$$

(Note that μ_m is the Laplace transform of $g(t)$.) By footnote 4,

$$g(t) = \sum_{j=0}^{\infty} (-1)^j \frac{B_j^{(1/2)}(\alpha)}{j!} t^{j+p-1/2}, \quad |t| < 2\pi. \tag{2.14}$$

We now apply Watson's lemma to the integral in (2.13) and obtain (2.11). For Watson's lemma, see Olver [21], for example. ∎

Remark. An important fact to realize in the asymptotic expansion of μ_m is that the powers of m in (2.11) are *independent of* α. (The coefficients in this expansion do depend on α and p, but this is of no concern to us, as we will see later.)

Throughout the remainder of this work, we shall let

$$\zeta = 1 - z^2. \tag{2.15}$$

By Theorem 2.2, the sequence $\{\mu_m\}_{m=1}^{\infty}$ is in a class of sequences denoted by $\mathbf{A}_0^{(-p-1/2)}$ in [6, Chapter 6, Subsection 6.1.1]. Consequently, the sequence $\{\mu_m \zeta^{-m}\}_{m=1}^{\infty}$ is in the sequence class denoted $\mathbf{b}^{(1)}$ in [6, Chapter 6, Subsection 6.1.2]. In addition, by [6, Chapter 6, Theorems 6.7.2 and 6.7.3] (see also [22, Theorems 2.1 and 2.2]) and by the fact that μ_m is a Laplace transform due to (2.13) with (2.14), the partial sums of the infinite series $\sum_{k=1}^{\infty} \mu_k \zeta^{-k}$, namely,

$$S_m(\zeta) = \sum_{k=1}^{m} \mu_k \zeta^{-k}, \quad m = 1, 2, \ldots, \tag{2.16}$$

satisfy

$$S_{m-1}(\zeta) \sim G(\zeta) + \mu_m \zeta^{-m} \sum_{i=0}^{\infty} \beta_i m^{-i} \quad \text{as } m \to \infty,\ z \notin [-1, 1], \tag{2.17}$$

with

$$G(\zeta) = -\int_{-1}^{1} \frac{w(x)}{z^2 - x^2}\,dx = -z^{-1} H(z), \quad z \notin [-1, 1]. \tag{2.18}$$

Of course, by (2.8),

$$G(\zeta) = \sum_{k=1}^{\infty} \mu_k \zeta^{-k}, \quad \text{when } |\zeta| = |1 - z^2| > 1. \tag{2.19}$$

Naturally, the β_i in (2.17) depend on ζ, α, and p; they do not depend on m, however.

---

4 The generalized Bernoulli polynomials $B_s^{(\sigma)}(u)$ are defined via (see Andrews, Askey, and Roy [20, p. 615], for example)

$$\left(\frac{t}{e^t - 1}\right)^{\sigma} e^{ut} = \sum_{s=0}^{\infty} B_s^{(\sigma)}(u) \frac{t^s}{s!}, \quad |t| < 2\pi.$$

They satisfy $B_s^{(\sigma)}(\sigma - u) = (-1)^s B_s^{(\sigma)}(u)$; hence $B_s^{(\sigma)}(\sigma/2) = 0$ for $s = 1, 3, 5, \ldots$. $B_s^{(\sigma)}(0)$ are called the generalized Bernoulli numbers and are denoted by $B_s^{(\sigma)}$. Note that $B_0^{(\sigma)}(u) = 1$ and $B_0^{(\sigma)} = 1$ for all σ. In addition, $B_k^{(\sigma)}(u) = \sum_{s=0}^{k} \binom{k}{s} B_{k-s}^{(\sigma)} u^s$ for all k.

As explained in [1], an asymptotic expansion for $S_{m-1}(\zeta)$ different from that in (2.17) can be obtained by substituting the asymptotic expansion of μ_m given in Theorem 2.2 in (2.17) and by re-expanding in negative powers of m. Indeed, upon doing so, we obtain

$$S_{m-1}(\zeta) \sim G(\zeta) + m^{-p-1/2}\zeta^{-m}\sum_{i=0}^{\infty}\beta_i' m^{-i} \quad \text{as } m \to \infty, \tag{2.20}$$

which is also valid for all $z \notin [-1, 1]$. This asymptotic expansion is of the form given in (A.3) of the Appendix, with $A_m = S_m(\zeta), A = G(\zeta)$, and $\omega_m = m^{-p-1/2}\zeta^{-m}$ there.

Applying now the $\mathcal{L}$ transformation to the sequence $\{S_m(\zeta)\}$ via (A.5), we obtain as approximations to $G(\zeta)$,

$$A_n^{(j)}(\zeta) = \frac{\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(j+i+1)^{n+p-1/2}\zeta^{j+i}S_{j+i}(\zeta)}{\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(j+i+1)^{n+p-1/2}\zeta^{j+i}}, \tag{2.21}$$

which, for simplicity, we write in the form

$$A_n^{(j)}(\zeta) = \frac{\widehat{N}(\zeta)}{\widehat{D}(\zeta)} = \frac{\sum_{i=0}^{n}\lambda_i\zeta^{j+i}S_{j+i}(\zeta)}{\sum_{i=0}^{n}\lambda_i\zeta^{j+i}}; \quad \lambda_i = (-1)^{n-i}\binom{n}{i}(j+i+1)^{n+p-1/2}. \tag{2.22}$$

Note also that we define $S_0(\zeta) = 0$ when $j = 0$.

By (2.16), it is easy to see that $\zeta^m S_m(\zeta) = \sum_{k=1}^{m}\mu_k\zeta^{m-k}$ is a polynomial of degree $m-1$ and is nonzero at $\zeta = 0$. From this, it follows that $A_n^{(j)}(\zeta)$ is a rational function of ζ, with numerator $\widehat{N}(\zeta)$ of degree at most $j+n-1$ and denominator $\widehat{D}(\zeta)$ of degree exactly $j+n$. In addition, by the fact that $A_n^{(j)}(\zeta)$ is an approximation to $G(\zeta)$ and by (2.18), we have that

$$H_n^{(j)}(z) = -zA_n^{(j)}(\zeta) \tag{2.23}$$

is an approximation to $H(z)$. Clearly, $H_n^{(j)}(z)$ is a rational function of z, with numerator of degree at most $2j+2n-1$ and denominator of degree exactly $2j+2n$. Thus, we can use $H_n^{(j)}(z)$ to derive numerical quadrature formulas for $I[f]$.

2.1. An extension

Our approach applies equally well with the slightly more general weight function

$$w(x) = |x|^{\gamma}(1-x^2)^{\alpha}\,[\log(1-x^2)^{-1}]^p, \quad \alpha > -1,\ p > -(\gamma+1)/2. \tag{2.24}$$

For, in this case,

$$\mu_m = \int_0^{\infty} e^{-mt}g(t)\,dt, \quad g(t) = \left(\frac{t}{e^t-1}\right)^{1/2-\gamma/2} e^{(1/2-\gamma/2-\alpha)t}\,t^{p-1/2+\gamma/2}. \tag{2.25}$$

By footnote 4,

$$g(t) = \sum_{j=0}^{\infty}(-1)^j\frac{B_j^{(1/2-\gamma/2)}(\alpha)}{j!}t^{j+p-1/2+\gamma/2}, \quad |t| < 2\pi. \tag{2.26}$$

We now apply Watson's lemma to the integral in (2.25) and obtain

$$\mu_m \sim \sum_{j=0}^{\infty}(-1)^j\frac{B_j^{(1/2-\gamma/2)}(\alpha)}{j!}\,\frac{\Gamma(j+p+1/2+\gamma/2)}{m^{j+p+1/2+\gamma/2}} \quad \text{as } m \to \infty. \tag{2.27}$$

By this, it is easy to see that (2.20) can now be replaced by

$$S_{m-1}(\zeta) \sim G(\zeta) + m^{-p-1/2-\gamma/2}\zeta^{-m}\sum_{i=0}^{\infty}\beta_i' m^{-i} \quad \text{as } m \to \infty, \tag{2.28}$$

which is also valid for all $z \notin [-1, 1]$. This asymptotic expansion is of the form given in (A.3) with $A_m = S_m(\zeta), A = G(\zeta)$, and $\omega_m = m^{-p-1/2-\gamma/2}\zeta^{-m}$ there. Therefore, the $\mathcal{L}$ transformation can be applied as before to obtain numerical quadrature formulas whose abscissas are independent of α again.

Orthogonal polynomials with respect to the inner product $(F, G) = \int_{-1}^{1} w(x)F(x)G(x)\,dx$ with the weight function $w(x) = |x|^{\gamma}$ (i.e., $\alpha = 0$ and $p = 0$ in (2.24)) are considered in Szegő [18, pp. 59–60]. It is shown there that, for this case, the corresponding orthogonal polynomials are related to the Jacobi polynomials $P_n^{(0,\gamma\pm1/2)}(2x^2-1)$.

3. Numerical quadrature formulas

We now turn to the derivation of our quadrature formulas. We start with the following theorem concerning the zeros of the denominator polynomial of $A_n^{(j)}(\zeta)$.

Theorem 3.1. *The denominator polynomial* $\widehat{D}(\zeta) = \sum_{i=0}^{n} \lambda_i \zeta^{j+i}$ *of* $A_n^{(j)}(\zeta)$ *has a zero of order* j *at* $\zeta = 0$ *and* n *simple real zeros in* $(-1, 1)$.

Proof. We start by noting that $\widehat{D}(\zeta) = \zeta^j D_n^{(p-1/2,j)}(\zeta)$, where $D_n^{(\sigma,\beta)}(\zeta)$ are polynomials introduced originally in [1] and given as in

$$D_n^{(\sigma,\beta)}(\zeta) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (\beta + i + 1)^{n+\sigma} \zeta^i. \tag{3.1}$$

It is clear that $\zeta = 0$ is a zero of order j of $\widehat{D}(\zeta)$. Next, it has been shown in [14, Theorems 2.1 and 2.2] that $D_n^{(\sigma,\beta)}(\zeta)$ satisfies the biorthogonality property

$$\int_0^1 D_n^{(\sigma,\beta)}(\xi)(\log \xi^{-1})^{k+\sigma} \xi^\beta \, d\xi = 0, \quad k = 0, 1, \ldots, n-1; \ \sigma, \beta > -1, \tag{3.2}$$

and hence that it has n simple real zeros in $(0, 1)$ since $\{(\log x^{-1})^k\}_{k=0}^{n-1}$ is an n-dimensional Chebyshev system on $(0, 1)$ and $\xi^\beta (\log x^{-1})^\sigma > 0$ on $(0, 1)$. The result now follows. ∎

The next theorem concerns the partial fraction decompositions of $A_n^{(j)}(\zeta)$ and $H_n^{(j)}(z)$.

Theorem 3.2. *Denote the* n *real zeros of* $\widehat{D}(\zeta)$, *the denominator of* $A_n^{(j)}(\zeta)$ *in (2.22), that are in* $(0, 1)$ *by* $\xi_{ni}^{(j)}$, $i = 1, \ldots, n$. *Then* $A_n^{(j)}(\zeta)$ *has a partial fraction decomposition of the form*

$$A_n^{(j)}(\zeta) = \sum_{i=0}^{j-1} \frac{\widehat{u}_{ni}^{(j)}}{\zeta^{i+1}} + \sum_{i=1}^{n} \frac{u_{ni}^{(j)}}{\zeta - \xi_{ni}^{(j)}}. \tag{3.3}$$

Consequently, $H_n^{(j)}(z)$ *has a partial fraction decomposition of the form*

$$H_n^{(j)}(z) = \sum_{i=0}^{j-1} \widehat{w}_{ni}^{(j)} \left[\frac{1}{(z-1)^{i+1}} + \frac{(-1)^i}{(z+1)^{i+1}} \right] + \sum_{i=1}^{n} w_{ni}^{(j)} \left[\frac{1}{z - x_{ni}^{(j)}} + \frac{1}{z + x_{ni}^{(j)}} \right], \tag{3.4}$$

where $x_{ni}^{(j)}$ *are distinct and*

$$x_{ni}^{(j)} = \sqrt{1 - \xi_{ni}^{(j)}} \in (0, 1), \quad i = 1, \ldots, n. \tag{3.5}$$

In addition,

$$w_{ni}^{(j)} = \frac{u_{ni}^{(j)}}{2}, \quad i = 1, \ldots, n, \ \textit{for all } j, \tag{3.6}$$

and

$$\widehat{w}_{n0}^{(1)} = \frac{\widehat{u}_{n0}^{(1)}}{2} \quad \textit{for } j = 1. \tag{3.7}$$

Proof. Because $\widehat{D}(\zeta)$ has j zeros at $\zeta = 0$ and n real simple zeros in $(0, 1)$, it is clear that $A_n^{(j)}(\zeta)$ has a partial fraction decomposition of the form given in (3.3).

As for $H_n^{(j)}(z)$, we first note that it is an odd function of z since $A_n^{(j)}(\zeta)$ is an even function of z. This implies that it has a partial fraction decomposition of the form given in (3.4). The results in (3.6) and (3.7) follow from the fact

$$-\frac{z}{\zeta - \xi_{ni}^{(j)}} = \frac{1}{2} \left[\frac{1}{z - x_{ni}^{(j)}} + \frac{1}{z + x_{ni}^{(j)}} \right].$$

We leave the details to the reader. ∎

We now use the partial fraction decomposition of $H_n^{(j)}(z)$ given in Theorem 3.2 to define our quadrature formulas:

- When $j = 0$, we have the formula

$$I_n^{(0)}[f] = \sum_{i=1}^{n} w_{ni}^{(0)}\big[f(x_{ni}^{(0)}) + f(-x_{ni}^{(0)})\big]. \tag{3.8}$$

- When $j = 1$, we have the formula

$$I_n^{(1)}[f] = \sum_{i=0}^{n} w_{ni}^{(1)}\big[f(x_{ni}^{(1)}) + f(-x_{ni}^{(1)})\big]; \quad x_{n0}^{(1)} = 1,\ w_{n0}^{(1)} = \widehat{w}_{n0}^{(1)}. \tag{3.9}$$

- For arbitrary j, we have the formula

$$I_n^{(j)}[f] = \sum_{i=0}^{j-1} \widehat{w}_{ni}^{(j)}\left[\frac{f^{(i)}(1)}{i!} + (-1)^i\frac{f^{(i)}(-1)}{i!}\right] + \sum_{i=1}^{n} w_{ni}^{(j)}\big[f(x_{ni}^{(j)}) + f(-x_{ni}^{(j)})\big]. \tag{3.10}$$

Here, $f^{(i)}(x)$ is the ith derivative of $f(x)$.

Remarks. 1. The quadrature formula $I_n^{(j)}[f]$ has $2j + 2n$ abscissas. Of these, $x = x_{ni}^{(j)}$ and $x = -x_{ni}^{(j)}$, $i = 1, \dots, n$ are simple ($2n$ in number), while $x = 1$ and $x = -1$ are j-fold each ($2j$ in number, counting multiplicities).
2. The formula $I_n^{(0)}[f]$ in (3.8) has all its abscissas in $(-1, 1)$ just like the corresponding Gaussian formula.
3. The formula $I_n^{(1)}[f]$ in (3.9), in addition to having $2n$ of its abscissas in $(-1, 1)$, has also the endpoints $x = \pm 1$ as abscissas, just like the corresponding Lobatto formula.

Note that, by the fact that $\xi_{ni}^{(j)}$ are *simple* poles of $A_n^{(j)}(\zeta) = \widehat{N}(\zeta)/\widehat{D}(\zeta)$, we have

$$u_{ni}^{(j)} = \mathrm{Res} A_n^{(j)}(\zeta)\big|_{\zeta=\xi_{ni}^{(j)}} = \frac{\widehat{N}(\xi_{ni}^{(j)})}{\widehat{D}'(\xi_{ni}^{(j)})}, \qquad \widehat{D}'(\zeta) = \frac{d}{d\zeta}\widehat{D}(\zeta),$$

by (3.3). Consequently, by (3.6), in all the formulas given in (3.8)–(3.10),

$$w_{ni}^{(j)} = \frac{1}{2}\frac{\widehat{N}(\xi_{ni}^{(j)})}{\widehat{D}'(\xi_{ni}^{(j)})} = \frac{1}{2}\left.\frac{\sum_{i=0}^{n}\lambda_i\zeta^{j+i}S_{j+i}(\zeta)}{\sum_{i=0}^{n}(j+i)\lambda_i\zeta^{j+i-1}}\right|_{\zeta=\xi_{ni}^{(j)}}. \tag{3.11}$$

Note also that, because the denominator polynomials $\widehat{D}(\zeta)$ of the $A_n^{(j)}(\zeta)$ are independent of α, so are their zeros $\zeta_{ni}^{(j)}$ and so are the abscissas $\pm x_{ni}^{(j)}$ in the quadrature formulas above. The $\pm x_{ni}^{(j)}$ do depend on p, however. The weights $w_{ni}^{(j)}$ and $\widehat{w}_{ni}^{(j)}$ depend on both α and p through the $S_m(\zeta)$ via the μ_m. (At the end of Section 4, we show when and how the $\pm x_{ni}$ can be made independent of p as well.)

We close this section with the following observation that is useful when checking tables of abscissas and weights:

Theorem 3.3. *Let*

$$g(x, z) = \frac{1}{z - x}. \tag{3.12}$$

Then the numerical quadrature formulas derived above satisfy

$$I_n^{(j)}[g(\cdot, z)] = H_n^{(j)}(z) = -zA_n^{(j)}(\zeta), \quad \zeta = 1 - z^2,\ z \notin [-1, 1]. \tag{3.13}$$

The proof of this theorem follows from (2.23), (3.4) and (3.10). We leave the details to the reader. The result in (3.13) can be used to compute the rational function $H_n^{(j)}(z)$, for arbitrary z, once via the quadrature formula $I_n^{(j)}[g(\cdot, z)]$ and once (and accurately) via $-zA_n^{(j)}(\zeta)$ given by the Levin transformation. Comparison of the numbers thus obtained can give us an indication about the correctness and/or accuracy of $x_{ni}^{(j)}$ and $w_{ni}^{(j)}$, the abscissas and weights of the quadrature formula $I_n^{(j)}$.

4. Properties of $I_n^{(j)}[f]$

The quadrature formulas we have just derived have some rather interesting properties, which we explore below.

Theorem 4.1. *$I_n^{(j)}[f]$ is interpolatory, that is,*

$$I_n^{(j)}[f] = I[f] \quad \text{for all } f \in \Pi_{2j+2n-1}, \tag{4.1}$$

where Π_m denotes the set of polynomials of degree at most m.

Proof. It is sufficient to show that (4.1) holds for $f = \phi_k$, $1 \le k \le 2j + 2n$, where $\phi_k(x) = x^{k-1}$, $k \ge 1$. For simplicity of notation we will let $\widehat{w}_i$, w_i, and x_i stand for $\widehat{w}_{ni}^{(j)}$, $w_{ni}^{(j)}$, and $x_{ni}^{(j)}$, respectively. The proof proceeds through three steps:

(i) First, it is easy to show that, for $k = 1, 2, \ldots,$

$$I_n^{(j)}[\phi_k] = \left\{ \sum_{i=0}^{\langle j,k \rangle} \widehat{w}_i \binom{k-1}{i} + \sum_{i=1}^{n} w_i x_i^{k-1} \right\} [1 + (-1)^{k-1}]; \quad \langle j, k \rangle = \min\{j-1, k-1\}. \tag{4.2}$$

(ii) Next, by expanding the partial fraction decomposition of $H_n^{(j)}(z)$ given in (3.4) in negative powers of z with $|z| > 1$, we obtain the convergent expansion

$$H_n^{(j)}(z) = \sum_{i=0}^{j-1} \widehat{w}_i \left\{ \sum_{s=0}^{\infty} \binom{-i-1}{s} [(-1)^i + (-1)^s] z^{-s-i-1} \right\} + \sum_{i=1}^{n} w_i \left\{ \sum_{s=0}^{\infty} [1 + (-1)^s] x_i^s z^{-s-1} \right\}. \tag{4.3}$$

Letting $k = s + i + 1$ in the first term and $k = s + 1$ in the second term, and changing the orders of summation, we obtain

$$H_n^{(j)}(z) = \sum_{k=1}^{\infty} \left\{ \sum_{i=0}^{\langle j,k \rangle} \widehat{w}_i [(-1)^i + (-1)^{k-i-1}] \binom{-i-1}{k-i-1} \right\} z^{-k} + \sum_{k=1}^{\infty} \left\{ \sum_{i=1}^{n} w_i [1 + (-1)^{k-1}] x_i^{k-1} \right\} z^{-k}. \tag{4.4}$$

Now,

$$(-1)^{k-i-1} \binom{-i-1}{k-i-1} = \binom{k-1}{k-i-1} = \binom{k-1}{i}.$$

Consequently, (4.4) can be re-expressed as in

$$H_n^{(j)}(z) = \sum_{k=1}^{\infty} \left\{ \sum_{i=0}^{\langle j,k \rangle} \widehat{w}_i \binom{k-1}{i} + \sum_{i=1}^{n} w_i x_i^{k-1} \right\} [1 + (-1)^{k-1}] z^{-k},$$

which, upon invoking (4.2), becomes

$$H_n^{(j)}(z) = \sum_{k=1}^{\infty} I_n^{(j)}[\phi_k] \, z^{-k}, \quad |z| > 1. \tag{4.5}$$

Of course, the summation $\sum_{k=1}^{\infty} I_n^{(j)}[\phi_k] \, z^{-k}$ is also the asymptotic expansion of $H_n^{(j)}(z)$ as $z \to \infty$.

(iii) Next, subtracting $H_n^{(j)}(z) = -zA_n^{(j)}(\zeta)$ from $H(z) = -zG(\zeta)$, with $A_n^{(j)}(\zeta)$ as in (2.22), we obtain

$$H(z) - H_n^{(j)}(z) = -z \frac{\sum_{i=0}^{n} \lambda_i \zeta^{j+i} [G(\zeta) - S_{j+i}(\zeta)]}{\sum_{i=0}^{n} \lambda_i \zeta^{j+i}}. \tag{4.6}$$

Now, by (2.19), we have $G(\zeta) - S_m(\zeta) = O(\zeta^{-m-1})$ as $\zeta \to \infty$. Consequently, the numerator of (4.6) is $O(\zeta^{-1})$ as $\zeta \to \infty$. In addition, the denominator is asymptotically equal to $\lambda_n \zeta^{j+n}$ as $\zeta \to \infty$. As a result,

$$H(z) - H_n^{(j)}(z) = O(z\zeta^{-j-n-1}) = O(z^{-2j-2n-1}) \quad \text{as } z \to \infty. \tag{4.7}$$

Here we have used the fact that z and ζ tend to infinity simultaneously and that $\zeta \sim -z^2$ as $z \to \infty$.

Substituting the convergent expansion $(z - x)^{-1} = \sum_{k=1}^{\infty} x^{k-1} z^{-k}$ with $|z| > 1$ in (1.5), we obtain the convergent expansion

$$H(z) = \sum_{k=1}^{\infty} \widehat{\mu}_k z^{-k}; \quad \widehat{\mu}_k = \int_{-1}^{1} w(x) x^{k-1} \, dx = I[\phi_k], \quad k = 1, 2, \ldots,$$

and thus

$$H(z) = \sum_{k=1}^{\infty} I[\phi_k] \, z^{-k}, \quad |z| > 1. \tag{4.8}$$

Of course, the summation $\sum_{k=1}^{\infty} I[\phi_k] \, z^{-k}$ is also the asymptotic expansion of $H(z)$ as $z \to \infty$.

Substituting (4.5) and (4.8) in (4.7), we then have

$$I_n^{(j)}[\phi_k] = I[\phi_k], \quad k = 1, \ldots, 2j + 2n.$$

This completes the proof. ∎

Remark. In some instances, $f^{(i)}(\pm 1), 0 \le i \le j-1$ for some $j > 0$ may be readily available. Then, in view of the result of Theorem 4.1, it is more advantageous to employ the formula $I_n^{(j)}[f]$ rather than $I_n^{(0)}[f]$ since the former integrates more polynomials exactly than the latter.

Our next result concerns the biorthogonality property of the polynomials

$$R_n^{(j)}(x) = (x^2-1)^j \prod_{i=1}^{n}(x^2-x_i^2), \tag{4.9}$$

where x_i again denotes $x_{ni}^{(j)}$ for short. Note that the abscissas of the quadrature formula $I_n^{(j)}[f]$ (with their multiplicities) are the zeros of $R_n^{(j)}(x)$.

Theorem 4.2. *The polynomials $R_n^{(j)}(x)$ have the biorthogonality property*

$$\int_{-1}^{1} R_n^{(j)}(x)\big[\log(1-x^2)^{-1}\big]^{p-1/2+k}|x|\,dx = 0, \quad k = 0, 1\ldots, n-1. \tag{4.10}$$

Proof. Recalling that $\widehat{D}(\zeta) = \zeta^j D_n^{(p-1/2,j)}(\zeta)$ and that $D_n^{(p-1/2,j)}(\zeta)$ satisfies (3.2), we have the biorthogonality property

$$\int_0^1 D_n^{(p-1/2,j)}(\xi)[\log \xi^{-1}]^{p-1/2+k}\xi^j d\xi = 0, \quad k = 0, 1, \ldots, n-1. \tag{4.11}$$

Making the change of variable $\xi = 1-x^2$, (4.11) becomes

$$\int_0^1 (1-x^2)^j D_n^{(p-1/2,j)}(1-x^2)\,[\log(1-x^2)^{-1}]^{p-1/2+k}x\,dx = 0, \quad k = 0, 1, \ldots, n-1. \tag{4.12}$$

The result now follows by recalling that $\widehat{D}(\xi) = \lambda_n \xi^j \prod_{i=1}^n(\xi - \xi_{ni}^{(j)})$ and that $\xi_{ni}^{(j)} = 1-[x_{ni}^{(j)}]^2$, from which we have

$(-1)^{j+n}\lambda_n^{-1}(1-x^2)^j D_n^{(p-1/2,j)}(1-x^2) = R_n^{(j)}(x)$. ∎

As a corollary of Theorems 4.1 and 4.2, we also have the following interesting result.

Theorem 4.3. *The quadrature formula $I_n^{(j)}[f]$ is exact for functions $f(x)$ that are of the form*

$$f(x) = \sum_{i=0}^{2j+2n-1} c_i x^i + |x|(1-x^2)^{-\alpha}R_n^{(j)}(x)\sum_{k=0}^{n-1} d_k\big[\log(1-x^2)^{-1}\big]^{k-1/2}, \tag{4.13}$$

d_k being arbitrary constants.

The case $p = 0, 1, 2, \ldots$

So far, we have seen that the quadrature formulas that we have developed for the weight function $w(x) = (1-x^2)^\alpha [\log(1-x^2)^{-1}]^p$ have abscissas that are independent of α. They do seem to depend on p, however. In case, p is a small integer such as $0, 1, 2, \ldots,$ these abscissas can be made independent of p too. This can be achieved as follows: When p is a nonnegative integer, the asymptotic expansion in (2.20), which forms the foundation of the effectiveness of the $\mathcal{L}$ transformation, can be rewritten in the form

$$S_{m-1}(\zeta) \sim G(\zeta) + m^{-1/2}\zeta^{-m}\sum_{i=0}^{\infty}\beta_i'' m^{-i} \quad \text{as } m \to \infty, \tag{4.14}$$

where

$$\beta_i'' = 0, \quad i = 0, 1, \ldots, p-1; \qquad \beta_i'' = \beta_{i-p}', \quad i = p, p+1, \ldots. \tag{4.15}$$

Applying the $\mathcal{L}$ transformation to the sequence $\{S_m(\zeta)\}$, but with $\omega_m = m^{-1/2}\zeta^{-m}$ now, we obtain

$$A_n^{(j)}(\zeta) = \frac{\widehat{N}(\zeta)}{\widehat{D}(\zeta)} = \frac{\sum_{i=0}^{n}\lambda_i\zeta^{j+i}S_{j+i}(\zeta)}{\sum_{i=0}^{n}\lambda_i\zeta^{j+i}}; \quad \lambda_i = (-1)^{n-i}\binom{n}{i}(j+i+1)^{n-1/2}. \tag{4.16}$$

From here on, we proceed exactly as before.

Note that, because the denominator polynomials $\widehat{D}(\zeta)$ of the $A_n^{(j)}(\zeta)$ are now independent of both α and p, so are their zeros $\zeta_{ni}^{(j)}$ and so are the abscissas $\pm x_{ni}^{(j)}$ in the quadrature formulas above. As before, the weights $w_{ni}^{(j)}$ and $\widehat{w}_{ni}^{(j)}$ depend on both α and p through the $S_m(\zeta)$ via the μ_m.

5. Integral representations for the weights $w_{ni}^{(0)}$

As we have seen, the abscissas $\pm x_{ni}$ and the weights $w_{ni}^{(0)}$ of the quadrature formula $I_n^{(0)}[f] = \sum_{i=1}^n w_{ni}^{(0)}[f(x_{ni}^{(0)})+f(-x_{ni}^{(0)})]$ are given by

$$x_{ni}^{(0)} = \sqrt{1-\xi_{ni}^{(0)}}, \qquad w_{ni}^{(0)} = \tfrac{1}{2}u_{ni}^{(0)}, \tag{5.1}$$

$\xi_{ni}^{(0)}$ being the poles of $A_n^{(0)}(\zeta)$ (hence zeros of the polynomial $D^{(-1/2,0)}(\zeta)$) and the $u_{ni}^{(0)}$ being the corresponding residues. We next provide an integral representation for the weights of an interpolatory quadrature formula involving an arbitrary symmetric weight function $W(x)$ and $2n$ abscissas $\pm x_i$.

Theorem 5.1. *Let $W(x)$ defined on $(-a, a)$ be such that $W(-x) = W(x)$ there, and let*

$$\hat{I}_n[f] = \sum_{i=1}^n \big[W_i f(x_i) + W_{-i} f(-x_i)\big]$$

be the interpolatory $2n$-point numerical quadrature formula for the integral $\hat{I}[f] = \int_{-a}^{a} W(x)f(x)dx$. Then

$$W_{-i} = W_i = \frac{1}{2}\int_{-a}^{a} W(x)\Bigg(\prod_{\substack{k=1\\k\neq i}}^{n} \frac{x^2-x_k^2}{x_i^2-x_k^2}\Bigg)dx, \quad i=1,\dots,n. \tag{5.2}$$

Proof. Using the fundamental polynomials of Lagrange, we first have

$$\begin{aligned} W_{\pm i} &= \int_{-a}^{a} W(x)\Bigg(\prod_{\substack{k=1\\k\neq i}}^{n} \frac{x-x_k}{\pm x_i - x_k}\Bigg)\Bigg(\prod_{\substack{k=1\\k\neq i}}^{n} \frac{x+x_k}{\pm x_i + x_k}\Bigg)\frac{x-(\mp x_i)}{\pm x_i - (\mp x_i)}\,dx \\ &= \int_{-a}^{a} W(x)\Bigg(\prod_{\substack{k=1\\k\neq i}}^{n} \frac{x^2-x_k^2}{x_i^2-x_k^2}\Bigg)\frac{x\pm x_i}{\pm 2x_i}\,dx. \end{aligned} \tag{5.3}$$

That $W_{-i} = W_i$ can be shown by making the variable transformation $t = -x$ in the integral representation of W_{-i} in (5.3) and by invoking $W(-x) = W(x)$. The integral representation in (5.2) is obtained by summing the integral representations of W_{-i} and W_i in (5.3) and dividing by 2. ■

Corollary 5.2. *Let $W(x) = U(1-x^2)$ and denote $\xi = 1-x^2$ and $\xi_i = 1-x_i^2, i = 1,\dots,n$. Let also $W_{\pm i}$ be the weights associated with the interpolatory quadrature formula*

$$\hat{I}_n[f] = \sum_{i=1}^n \big[W_i f(x_i) + W_{-i} f(-x_i)\big]$$

for the integral $\hat{I}[f] = \int_{-1}^{1} W(x)f(x)dx$. Then

$$W_{\pm i} = \frac{1}{2}\int_0^1 \frac{U(\xi)}{\sqrt{1-\xi}}\, l_i(\xi)\,d\xi, \quad i=1,\dots,n,$$

where $l_i(\xi)$ are the fundamental polynomials of Lagrange over the set of points $\{\xi_1,\dots\xi_n\}$ given as in

$$l_i(\xi) = \prod_{\substack{k=1\\k\neq i}}^{n} \frac{\xi-\xi_k}{\xi_i-\xi_k}, \quad i=1,\dots,n.$$

Corollary 5.3. *The weights $w_{ni}^{(0)}$ of the quadrature formula $I_n^{(0)}[f]$ obtained in Section 3 are given as in*

$$w_{ni}^{(0)} = \frac{1}{2}\int_0^1 \frac{U(\xi)}{\sqrt{1-\xi}}\,\frac{D_n(\xi)}{D_n'(\xi_{ni}^{(0)})}\,\frac{d\xi}{\xi-\xi_{ni}^{(0)}}, \quad i=1,\dots,n. \tag{5.4}$$

Here $U(\xi) = \xi^\alpha(\log\xi^{-1})^p$, $D_n(\xi) = D_n^{(p-1/2,0)}(\xi)$, and $D_n'(\xi) = \frac{d}{d\xi}D_n(\xi)$. Invoking (4.11), we also have

$$w_{ni}^{(0)} = \frac{1}{2}\int_0^1 (\log\xi^{-1})^{p-1/2}\frac{D_n(\xi)}{D_n'(\xi_{ni}^{(0)})}\Bigg[\frac{\xi^\alpha(\log\xi^{-1})^{1/2}}{\sqrt{1-\xi}}\,\frac{1}{\xi-\xi_{ni}^{(0)}} - \sum_{k=0}^{n-1} c_k(\log\xi^{-1})^k\Bigg]d\xi, \tag{5.5}$$

$c_0, c_1, \dots, c_{n-1}$ being arbitrary constants.

By choosing the c_k in (5.5) appropriately, it might be possible to show that the weights $w_{ni}^{(0)}$ in the quadrature formula $I_n^{(0)}$ are positive for all n, at least for some cases. This approach was used successfully in Lubinsky and Sidi [11] for some of the quadrature formulas of Sidi [1]. So far, we have not been able to prove such a result for the quadrature formulas of this work, however.

6. Computation of tables and numerical examples

6.1. Computation of tables

We have computed the abscissas and weights for the quadrature formulas $I_n^{(0)}[f] \equiv I_n[f]$ for the cases of $\alpha = 0, \pm\frac{1}{2}$ with $p = 0, 1$. Thus, these $2n$-point quadrature formulas are given as in

$$I_n[f] = \sum_{i=1}^{n} w_{ni}[f(x_{ni}) + f(-x_{ni})],$$

with

$$x_{ni} = \sqrt{1-\xi_{ni}}, \qquad w_{ni} = \tfrac{1}{2}u_{ni}, \quad i = 1, \ldots, n,$$

where ξ_{ni} are the zeros of the polynomial $\widehat{D}(\xi) = D^{(-1/2,0)}(\xi)$, that is,

$$\widehat{D}(\xi) = \sum_{i=0}^{n} \lambda_i \xi^i, \quad \lambda_i = (-1)^{n-i}\binom{n}{i}(i+1)^{n-1/2}, \ i = 0, 1, \ldots, n,$$

while u_{ni} are the residues of the rational function

$$G_n(\zeta) = \frac{\widehat{N}(\zeta)}{\widehat{D}(\zeta)} = \frac{\sum_{i=0}^{n} \lambda_i \zeta^i S_i(\zeta)}{\sum_{i=0}^{n} \lambda_i \zeta^i}, \qquad S_i(\zeta) = \sum_{m=1}^{i} \mu_m \zeta^{-m}, \qquad S_0(\zeta) \equiv 0,$$

that is

$$u_{ni} = \frac{\sum_{k=0}^{n} \lambda_k \xi_{ni}^k S_k(\xi_{ni})}{\sum_{k=1}^{n} k\lambda_k \xi_{ni}^{k-1}}, \quad i = 1, \ldots, n.$$

We would like to emphasize that the abscissas used by all the quadrature formulas mentioned here are the same.

These abscissas, along with the weights for the case $(\alpha, p) = (0, 0)$, are given in Table A.1. Note that because the polynomials $\widehat{D}(\xi)$ are known explicitly, we can use any polynomial solver to determine their zeros, namely, the ξ_{ni}. However, for large n, the computation of these zeros in finite-precision (floating-point) arithmetic to machine accuracy becomes difficult, the apparent reason being that the coefficients λ_i of the polynomial $\widehat{D}(\xi)$ have widely differing orders of magnitude. This suggests that, for large n, the zeros of $\widehat{D}(\xi)$ can be determined with a desired level of accuracy by using variable-precision arithmetic. We have done all our computations in quadruple-precision arithmetic (approximately 35 decimal digits).

Below, we give the μ_m and the $H(z)$ corresponding to the weight function $w(x) = (1-x^2)^\alpha[\log(1-x^2)^{-1}]^p$ for $\alpha = 0, \pm\frac{1}{2}$ and $p = 0, 1$; we make use of all these in our numerical examples in the next subsection. In our derivations, we have used the following facts about the Beta function $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ and the Psi function $\psi(z) = \Gamma'(z)/\Gamma(z)$, which can be found in Olver et al. [19], for example:

$$\int_{-1}^{1}(1-x^2)^{s-1}\,dx = B\left(s, \tfrac{1}{2}\right) = \frac{\Gamma(s)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(s+\frac{1}{2}\right)},$$

$$\psi(n+1) = -C + \sum_{k=1}^{n}\frac{1}{k}, \qquad \psi\left(n+\tfrac{1}{2}\right) = -C - 2\log 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1}, \quad n = 0, 1, \ldots,$$

where $C = 0.577\cdots$ is Euler's constant.

6.1.1. The case $p = 0$: $w(x) = (1-x^2)^\alpha$

For general α, we have

$$\mu_m = \int_{-1}^{1}(1-x^2)^{\alpha+m-1}\,dx = B\left(\alpha+m, \tfrac{1}{2}\right) = \frac{\Gamma(\alpha+m)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+m+\frac{1}{2})}$$

Table A.1

Abscissas and weights for $I_n[f] = \sum_{i=1}^{n} w_{ni}\Big[f(x_{ni}) + f(-x_{ni})\Big]$, where $I_n[f] \approx I[f] = \int_{-1}^{1} f(x)\,dx$.

x_{ni}	w_{ni}
$n = 1$	
5.411961001461969843997232 1D−01	1.000000000000000000000000D+00
$n = 2$	
8.819981412484646764581878 9D−01	3.101862857540213830254952 4D−01
3.652631529878405941313728 1D−01	6.898137142459786169745047 6D−01
$n = 3$	
9.629100345143411054828527 2D−01	1.104378701846527514709960 3D−01
7.318049301480553322895315 0D−01	3.555965374451249446781477 7D−01
2.754050485309774788824779 4D−01	5.339655923702223038508562 0D−01
$n = 4$	
9.868474166452178449947405 2D−01	4.408277181028374593051080 9D−02
8.770522995610385368390617 7D−01	1.847413905337251222798110 2D−01
6.134457219672696539709536 6D−01	3.378720793115240927947206 6D−01
2.209273616544173488283113 4D−01	4.333037583444670389949575 1D−01
$n = 5$	
9.949649339731304702844919 0D−01	1.888997864041431569390454 4D−02
9.397680938847072589871555 7D−01	1.000127787313915750280473 0D−01
7.853594775361605413719860 8D−01	2.098291431755069806019284 3D−01
5.243232569315763079673790 1D−01	3.074159449449787007958144 8D−01
1.843968754526927624542598 1D−01	3.638521545077084278803052 4D−01
$n = 6$	
9.979703115406346292604993 0D−01	8.464744695211864980950781 8D−03
9.689741347213591798350750 9D−01	5.623054566511509054719236 9D−02
8.760216249132287902330919 0D−01	1.322578035922164940336342 6D−01
7.031337438451115834050163 4D−01	2.124227175526120836778614 9D−01
4.563321077886213859170593 7D−01	2.773132591241434477482661 2D−01
1.582106972287487303515091 2D−01	3.133109293707010190120949 8D−01
$n = 7$	
9.991513180012153253500264 4D−01	3.907010026810933332789366 1D−03
9.833963641008008491843161 2D−01	3.258089441239222674895356 9D−02
9.259538874085426761175983 5D−01	8.509868385552234609229870 3D−02
8.100562472974031699829737 5D−01	1.471161238665251031153101 8D−01
6.329048269594251675468998 6D−01	2.056778443293955697172108 9D−01
4.032637240210066868381959 8D−01	2.506516683881766079068936 3D−01
1.385254491534673818356374 8D−01	2.749677751211772130865436 7D−01
$n = 8$	
9.996354511218560186032027 6D−01	1.840496684323385100022074 4D−03
9.908470760668937382052476 4D−01	1.932634901470097587482514 3D−02
9.545424845231529972727924 8D−01	5.583931773674143227187894 6D−02
8.756373076272430571872189 3D−01	1.029682987737309806126870 7D−01
7.480100324770056136749190 0D−01	1.519355743763309360105311 2D−01
5.736084097673496518299826 5D−01	1.954292010718607590826676 5D−01
3.608976384597514953816763 6D−01	2.277429102156470570125900 5D−01
1.231903002958224489736948 1D−01	2.449178521266644740347979 5D−01
$n = 9$	
9.998401770264870461600193 0D−01	8.795836636713899389950653 3D−04
9.948342819377422523992656 4D−01	1.167756816239077262484642 9D−02
9.714552194002258045037626 9D−01	3.727239358947295827530741 8D−02
9.169189434716761584730042 9D−01	7.296770067439367613681766 4D−02
8.241700447310354349636119 5D−01	1.127404659305165251066504 7D−01
6.918554158877286830370134 4D−01	1.512916445461537328896931 3D−01
5.234645699574897070742464 7D−01	1.842414674686177846291581 7D−01
3.263883302464249678268700 7D−01	2.081789679829623241002521 3D−01
1.109082056130568308881667 5D−01	2.207502079818208362982795 2D−01
$n = 10$	
9.999288091106440377886930 4D−01	4.247219410830415478433832 8D−04
9.970286077099534505937893 3D−01	7.160493563249794841977210 7D−03
9.817368673941701623488052 8D−01	2.524358621958272979841923 3D−02
9.435272204538894374081527 9D−01	5.233474101027185100042787 9D−02
8.754517982630701461625853 5D−01	8.428641657238442576421253 1D−02
7.746357691128203523778087 3D−01	1.172127707068565826450303 5D−01
6.418193433930997393176515 3D−01	1.477790700253492943554983 8D−01
4.807894546524732928773309 8D−01	1.732303377247861387459988 4D−01
2.977842097399879275259529 5D−01	1.914273246570645480162544 4D−01
1.008507601204819409932790 1D−01	2.009005375793715932843377 6D−01

(continued on next page)

Table A.1 (*continued*)

x_{ni}	w_{ni}
$n = 11$	
9.99967886117665751773215 25D−01	2.06648912630474231806433 40D−04
9.98263884349424626594976 41D−01	4.44319633774438462915383 16D−03
9.88130875901305408719913 69D−01	1.73081819826109250393568 60D−02
9.61038655237782984183963 72D−01	3.79509773029540028869552 29D−02
9.10582020324396458648133 44D−01	6.35380720159105369811512 29D−02
8.33278212339857471877298 31D−01	9.11897709218517715285809 06D−02
7.28369351372087130823003 64D−01	1.18353129095828531537354 39D−01
5.97467192239319974381149 26D−01	1.42843407844567146966847 99D−01
4.44179271815744915829022 73D−01	1.62857039621773409014284 92D−01
2.73715540681381130679003 77D−01	1.76998102089028059360175 17D−01
9.24641977661095642042790 83D−02	1.84311473875100757824333 05D−01
$n = 12$	
9.99985364848182813367539 571D−01	1.01119785548817885583928 59D−04
9.98972316291254182385987 94D−01	2.78405777996820208143705 88D−03
9.92183713386471562514951 30D−01	1.19912824745718070673571 28D−02
9.72773842534569786032553 31D−01	2.77908501615572165668004 16D−02
9.35034644924880724516156 23D−01	4.82887892427198499538402 19D−02
8.75348179643620688535572 08D−01	7.13378611321450170708139 19D−02
7.92168554885081651993321 67D−01	9.49662127658208080007752 76D−02
6.85802157124153114286079 90D−01	1.17443629510877635487940 47D−01
5.58161452620390392831833 52D−01	1.37289216456794201357803 38D−01
4.12510982307445666110726 54D−01	1.53282318049916533315915 55D−01
2.53198234937754551420274 75D−01	1.64481164153600318119627 08D−01
8.53643508826251050863838 34D−02	1.70243498486479593092105 67D−01

and

$$H(z) = B\left(\alpha+1, \tfrac{1}{2}\right) \frac{z}{z^2-1}\, {}_2F_1\left(\alpha+1, 1; \alpha+\tfrac{3}{2}; (1-z^2)^{-1}\right).$$

Here are the special cases of $\alpha = 0, \pm\frac{1}{2}$:

1. *The case* $\alpha = 0$

$$\mu_m = 2\frac{(m-1)!}{\left(\frac{3}{2}\right)_{m-1}}, \qquad H(z) = \log\frac{z+1}{z-1}.$$

2. *The case* $\alpha = -1/2$

$$\mu_m = \pi\frac{\left(\frac{1}{2}\right)_{m-1}}{(m-1)!}, \qquad H(z) = \frac{\pi}{(z^2-1)^{1/2}}.$$

3. *The case* $\alpha = 1/2$

$$\mu_m = \pi\frac{\left(\frac{1}{2}\right)_m}{m!}, \qquad H(z) = \pi\left[z-(z^2-1)^{1/2}\right].$$

6.1.2. *The case* $p = 1$: $w(x) = (1-x^2)^\alpha \log(1-x^2)^{-1}$

By the fact that

$$w(x) = (1-x^2)^\alpha \log(1-x^2)^{-1} = -\frac{\partial}{\partial\alpha}(1-x^2)^\alpha,$$

for general α, we have

$$\mu_m = -\frac{\partial}{\partial\alpha}B\left(\alpha+m, \tfrac{1}{2}\right) = \left[\psi(\alpha+m+\tfrac{1}{2}) - \psi(\alpha+m)\right]B\left(\alpha+m, \tfrac{1}{2}\right)$$

and

$$H(z) = -\frac{z}{z^2-1}\frac{\partial}{\partial\alpha}\left\{B\left(\alpha+1, \tfrac{1}{2}\right) {}_2F_1\left(\alpha+1, 1; \alpha+\tfrac{3}{2}; (1-z^2)^{-1}\right)\right\}.$$

Here are the special cases of $\alpha = 0, \pm\frac{1}{2}$:

1. *The case* $\alpha = 0$

$$\mu_m = 2\frac{(m-1)!}{\left(\frac{3}{2}\right)_{m-1}}\left(-2\log 2 + 2\sum_{k=m}^{2m-1}\frac{1}{k}\right),$$

$$H(z) = -z\sum_{m=0}^{\infty}\mu_m(1-z^2)^{-m}, \qquad |z^2-1| > 1.$$ [5]

2. *The case* $\alpha = -1/2$

$$\mu_m = \pi\frac{\left(\frac{1}{2}\right)_{m-1}}{(m-1)!}\left(2\log 2 - 2\sum_{k=m}^{2m-2}\frac{1}{k}\right),$$

$$H(z) = \frac{2\pi}{\sqrt{z^2-1}}\log\frac{z+\sqrt{z^2-1}}{\sqrt{z^2-1}}.$$

3. *The case* $\alpha = 1/2$

$$\mu_m = \pi\frac{\left(\frac{1}{2}\right)_m}{m!}\left(2\log 2 - 2\sum_{k=m+1}^{2m}\frac{1}{k}\right),$$

$$H(z) = (2\pi\log 2)z - 2\pi\sqrt{z^2-1}\,\log\frac{z+\sqrt{z^2-1}}{\sqrt{z^2-1}}.$$

The expression for $H(z)$ in case $\alpha = -\frac{1}{2}$ appears in Gradshteyn and Ryzhik [23, p. 562, Eq. (4.295.36)]. The expression for $H(z)$ in case $\alpha = \frac{1}{2}$ is obtained from

$$\begin{aligned}\int_{-1}^{1}\log(1-x^2)^{-1}\sqrt{1-x^2}\,\frac{dx}{z-x} &= \int_{-1}^{1}\frac{\log(1-x^2)^{-1}}{\sqrt{1-x^2}}\left(x+z+\frac{1-z^2}{z-x}\right)dx\\ &= z\int_{-1}^{1}\frac{\log(1-x^2)^{-1}}{\sqrt{1-x^2}}\,dx + (1-z^2)\int_{-1}^{1}\frac{\log(1-x^2)^{-1}}{\sqrt{1-x^2}}\,\frac{dx}{z-x},\end{aligned}$$

and from μ_1 and $H(z)$ of the case $\alpha = -\frac{1}{2}$.

6.2. Numerical examples

We have applied the new quadrature formulas discussed in the preceding subsection to several integrals and observed that they have excellent convergence properties. Here we bring some of the numerical results. All our computations have been carried out in quadruple-precision arithmetic (approximately 35 decimal digits).

Example 6.1. First, we have applied $I_n^{(0)} = I_n$ to the function $g(x,z) = 1/(z-x)$. In Table A.2, we give the relative errors in the approximations $I_n[g(\cdot,2)]$ to the integrals $I[g(\cdot,2)]$ with $g(x,z) = 1/(z-x)$ and for $\alpha = 0, \pm\frac{1}{2}$ and $p = 0, 1$ in the weight function, the abscissas of the quadrature formulas used being the same for all approximations. These integrals were considered in the preceding subsection. As mentioned following Theorem 3.3, for these integrals, we have

$$I_n[g(\cdot,z)] = H_n^{(0)}(z) = -zA_n^{(0)}(1-z^2), \quad z \notin [-1,1].$$

The errors shown in Table A.2 are actually those computed from the appropriate $-zA_n^{(0)}(1-z^2)$, the reason for this being that we would like to ascertain the *actual* efficiency of our quadrature formulas, since $A_n^{(0)}(\zeta)$ can be computed with machine accuracy. We would like to note that, due to the fact that we are not able to compute the abscissas, and hence also the weights, of the quadrature formulas $I_n^{(0)}$ for large n to machine precision, the approximations $I_n^{(0)}[g(\cdot,z)]$ via the quadrature formulas (with abscissas and weights computed in floating-point arithmetic) do not achieve machine accuracy for large n, even though they must be identical to $-zA_n^{(0)}(1-z^2)$ theoretically. (To achieve the same accuracy as $-zA_n^{(0)}(1-z^2)$, the abscissas and weights of $I_n^{(0)}$ must be computed to machine accuracy, preferably in variable-precision arithmetic, as we mentioned earlier.) In this respect, we would like to mention that, with $z = 2$ in $g(z,x)$, the smallest relative errors obtained from actual use of the quadrature formulas (computed using quadruple-precision arithmetic, as in Table A.1) are of order 10^{-23} and take place for $n \approx 13$ and increase gradually as n is increased due to the limited accuracy of the computed abscissas and weights. This is so for all the integrals treated in the present example. For $z = 5$, the smallest relative errors are of order 10^{-27} and take place for $n \approx 11$ for all the integrals.

[5] Unfortunately, at the present, we do not have a simple expression for $H(z)$ in this case.

Table A.2
Relative errors for the quadrature formulas $I_n^{(\alpha,p)} \equiv I_n^{(0)}$ of Section 6.2 as these are applied to the integrals $I^{(\alpha,p)}[g(\cdot,z)]$, $g(x,z) = 1/(z-x)$, where $I^{(\alpha,p)}[f] = \int_{-1}^{1}(1-x^2)^\alpha[\log(1-x^2)^{-1}]^p f(x)\,dx$. $E_n^{(\alpha,p)}$ stands for $\left|I_n^{(\alpha,p)}[g(\cdot,2)] - I^{(\alpha,p)}[g(\cdot,2)]\right| / \left|I^{(\alpha,p)}[g(\cdot,2)]\right|$. The abscissas in $I_n^{(\alpha,p)}$ are the same for all α and for $p = 0, 1$, and are determined as in Section 6.1.

n	$E_n^{(0,0)}$	$E_n^{(-1/2,0)}$	$E_n^{(1/2,0)}$	$E_n^{(0,1)}$	$E_n^{(-1/2,1)}$	$E_n^{(1/2,1)}$
1	1.78D−02	6.56D−02	6.73D−03	1.14D−01	1.56D−01	8.08D−02
2	7.24D−05	2.16D−03	1.30D−04	1.87D−03	8.46D−03	1.67D−03
3	5.28D−07	4.39D−05	5.14D−06	1.02D−05	2.42D−04	2.69D−06
4	3.33D−08	6.83D−07	7.17D−08	2.17D−08	4.63D−06	7.00D−07
5	7.65D−10	8.06D−09	9.19D−11	2.15D−09	6.74D−08	1.49D−08
6	9.34D−12	7.55D−11	2.84D−11	7.13D−11	7.66D−10	1.09D−10
7	3.77D−14	7.70D−13	7.18D−13	1.28D−12	7.28D−12	2.60D−12
8	4.87D−15	3.42D−15	8.53D−15	8.61D−15	6.44D−14	1.09D−13
9	1.22D−16	2.51D−17	4.58D−17	2.81D−16	3.42D−16	1.96D−15
10	1.43D−18	8.94D−19	4.64D−18	1.15D−17	3.27D−18	1.15D−17
11	9.52D−21	1.64D−20	1.10D−19	2.09D−19	3.43D−20	4.56D−19
12	8.40D−22	1.94D−22	1.21D−21	1.14D−21	6.65D−22	1.73D−20
13	1.94D−23	1.69D−24	1.07D−23	5.28D−23	1.32D−23	2.97D−22
14	2.01D−25	1.22D−25	7.83D−25	1.92D−24	5.53D−26	1.33D−24
15	2.22D−27	2.70D−27	1.74D−26	3.21D−26	3.53D−27	8.24D−26
16	1.42D−28	2.61D−29	1.69D−28	1.17D−28	1.21D−28	2.82D−27
17	3.06D−30	3.56D−31	2.30D−30	9.64D−30	1.93D−30	4.53D−29
18	2.80D−32	2.04D−32	1.32D−31	3.12D−31	5.12D−33	1.24D−31
19	7.01D−34	4.25D−34	2.86D−33	6.96D−33	5.53D−34	1.54D−32
20	1.75D−34	1.06D−34	1.14D−34	2.45D−33	0.00D+00	2.70D−34

Example 6.2. We have applied our quadrature formulas to the integrals $I[f]$, with $f(x) = 1/(1+x^2)$ and $\alpha = 0, \pm\frac{1}{2}$, $p = 0, 1$ in the weight function $w(x) = (1-x^2)^\alpha[\log(1-x^2)^{-1}]^p$. For these integrals, we have

$$(\alpha,p) = (0,0) \qquad I[f] = \frac{\pi}{2}$$
$$(\alpha,p) = \left(-\frac{1}{2},0\right) \qquad I[f] = \frac{\pi}{\sqrt{2}}$$
$$(\alpha,p) = \left(\frac{1}{2},0\right) \qquad I[f] = \pi\left(\sqrt{2}-1\right)$$
$$(\alpha,p) = (0,1) \qquad I[f] = 2G - \frac{\pi}{2}\log 2, \quad G = 0.915965594177\cdots \text{ Catalan's constant}$$
$$(\alpha,p) = \left(-\frac{1}{2},1\right) \qquad I[f] = \pi\sqrt{2}\log(1+1/\sqrt{2})$$
$$(\alpha,p) = \left(\frac{1}{2},1\right) \qquad I[f] = 2\pi[\sqrt{2}\log(1+1/\sqrt{2}) - \log 2]$$

The relative errors in the $I_n[f]$ are given in Table A.3. Again, the abscissas of the quadrature formulas used are the same for all approximations. Note that the errors are increasing towards the bottom of Table A.3 when, actually, they should be decreasing and should do so fast. Of course, the reason for this is that the abscissas and weights we are using here have not been computed to machine accuracy due to limitations imposed by the floating-point arithmetic used in determining them.

Acknowledgments

This research was supported by United States–Israel Binational Science Foundation grant no. 2008399. The author would like to thank Professor Doron S. Lubinsky for useful conversations he had with him during the preparation of this paper.

Appendix. Levin $\mathcal{L}$ transformation

The Levin [10] $\mathcal{L}$ transformation is a most successful method used for accelerating the convergence of infinite sequences $\{A_m\}$ whose members are such that

$$A_{m-1} = A + \omega_m h(m), \tag{A.1}$$

Table A.3

Relative errors for the quadrature formulas $I_n^{(\alpha,p)} \equiv I_n^{(0)}$ of Section 6.2 as these are applied to the integrals $I^{(\alpha,p)}[f], f(x) = 1/(1+x^2)$, where $I^{(\alpha,p)}[f] = \int_{-1}^{1}(1-x^2)^\alpha[\log(1-x^2)^{-1}]^p f(x)\,dx$. $E_n^{(\alpha,p)}$ stands for $\left|I_n^{(\alpha,p)}[f] - I^{(\alpha,p)}[f]\right| / \left|I^{(\alpha,p)}[f]\right|$. The abscissas in $I_n^{(\alpha,p)}$ are the same for all α and for $p = 0, 1$, and are determined as in Section 6.1.

n	$E_n^{(0,0)}$	$E_n^{(-1/2,0)}$	$E_n^{(1/2,0)}$	$E_n^{(0,1)}$	$E_n^{(-1/2,1)}$	$E_n^{(1/2,1)}$
1	1.52D−02	9.38D−02	6.64D−02	2.77D−01	4.18D−01	1.82D−01
2	2.95D−03	9.52D−03	4.82D−03	3.41D−03	4.43D−02	1.48D−02
3	2.48D−04	2.41D−04	4.04D−04	3.53D−04	2.87D−03	9.46D−04
4	2.06D−05	2.17D−05	3.49D−05	3.25D−05	1.07D−04	7.59D−05
5	1.75D−06	5.91D−07	3.01D−06	2.63D−06	4.01D−06	6.73D−06
6	1.51D−07	8.17D−08	2.58D−07	2.21D−07	3.49D−08	5.88D−07
7	1.30D−08	6.41D−09	2.23D−08	1.90D−08	5.94D−09	5.07D−08
8	1.12D−09	5.64D−10	1.92D−09	1.65D−09	3.35D−10	4.36D−09
9	9.71D−11	4.85D−11	1.66D−10	1.42D−10	3.19D−11	3.77D−10
10	8.39D−12	4.20D−12	1.43D−11	1.23D−11	2.71D−12	3.26D−11
11	7.26D−13	3.63D−13	1.24D−12	1.06D−12	2.35D−13	2.81D−12
12	6.27D−14	3.14D−14	1.07D−13	9.19D−14	2.03D−14	2.43D−13
13	5.43D−15	2.71D−15	9.26D−15	7.95D−15	1.70D−15	2.11D−14
14	4.69D−16	2.35D−16	8.01D−16	6.88D−16	9.50D−17	1.82D−15
15	4.06D−17	2.03D−17	6.93D−17	5.95D−17	4.39D−17	1.58D−16
16	3.49D−18	1.76D−18	6.02D−18	5.15D−18	5.59D−17	1.36D−17
17	3.30D−19	2.64D−19	6.30D−20	4.27D−19	5.70D−17	1.16D−18
18	9.39D−18	6.68D−18	7.27D−18	1.23D−18	5.68D−17	1.60D−19
19	2.68D−16	3.11D−16	4.16D−16	2.84D−18	5.92D−17	3.64D−17
20	4.11D−15	7.52D−15	1.61D−15	1.60D−16	2.47D−16	6.73D−16

where $h(m)$ is a function having an asymptotic expansion of the form

$$h(m) \sim \sum_{i=0}^{\infty} \frac{\beta_i}{m^i} \quad \text{as } m \to \infty. \tag{A.2}$$

Here A is either the limit of $\{A_m\}$ when the latter converges or the so-called *antilimit* of $\{A_m\}$ when the latter diverges.[6] Substituting (A.2) in (A.1), we have the asymptotic expansion

$$A_{m-1} \sim A + \omega_m \sum_{i=0}^{\infty} \frac{\beta_i}{m^i} \quad \text{as } m \to \infty. \tag{A.3}$$

We are only interested in determining (or approximating) A, whether it is the limit or the antilimit of $\{A_m\}$.

With the sequences $\{A_m\}$ and $\{\omega_m\}$ available, the Levin $\mathcal{L}$ transformation (based on the asymptotic expansion in (A.3)) is defined via the linear systems of equations

$$A_{m-1} = A_n^{(j)} + \omega_m \sum_{i=0}^{n-1} \frac{\bar{\beta}_i}{m^i}, \quad m = j+1, j+2, \ldots, j+n+1. \tag{A.4}$$

Here $j \geq 0$ and $n \geq 1$, and $A_n^{(j)}$ is the approximation to A, while $\bar{\beta}_i$ are additional (auxiliary) unknowns of no interest to us. The solution of (A.4) for $A_n^{(j)}$ can be expressed in closed form as in

$$A_n^{(j)} = \frac{\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(j+i+1)^{n-1}A_{j+i}/\omega_{j+i+1}}{\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(j+i+1)^{n-1}/\omega_{j+i+1}}. \tag{A.5}$$

Note that the linear system in (A.4) has been obtained from (A.3) by replacing A by $A_n^{(j)}$, β_i by $\bar{\beta}_i$, and the asymptotic equality sign $\sim$ by $=$, and by truncating the infinite series $\sum_{i=0}^{\infty}\beta_i/m^i$ at the $i = n-1$ term, and finally by collocating the equality obtained at the $n+1$ points $m = j+1, j+2, \ldots, j+n+1$, thus obtaining $n+1$ equations to accommodate the $n+1$ unknowns $A_n^{(j)}$ and $\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_{n-1}$. Also note that we do not need to know the β_i in (A.3); we would like to emphasize that mere knowledge of the *existence of* the asymptotic expansion in (A.3) together with the sequence $\{\omega_m\}$ is sufficient for applying the $\mathcal{L}$ transformation.

6 In case A_m is the mth partial sum of an infinite power series in z with a finite radius of convergence $\rho > 0$, $A = \lim_{m\to\infty} A_m$ for $|z| < \rho$ represents a function $f(z)$ that is analytic in the set $\{z : |z| < \rho\}$. If the function $f(z)$ can be continued analytically to some subset of $\{z : |z| \geq \rho\}$, then this analytic continuation is the *antilimit* of $\{A_m\}$ in this subset. For a discussion of antilimits and their various meanings, see [6, Introduction, Section 0.2], where several examples from different problems are also given.

In Levin's work [10], $\omega_m = m^\sigma(A_m - A_{m-1})$, where σ is some integer at most 1. When $\sigma = 0$, the $\mathcal{L}$ transformation is called the t-transformation, and when $\sigma = 1$, it is called the u-transformation. In developing our numerical quadrature formulas, however, we do not necessarily use Levin's ω_m; our ω_m are designed such that the asymptotic expansion in (A.3) is valid (with different β_i though) and the resulting quadrature formulas enjoy a great amount of flexibility. This approach was first suggested and used by the author in [1,2]. The convergence properties of the sequences $\{A_n^{(j)}\}_{j=0}^\infty$ (with n fixed) and $\{A_n^{(j)}\}_{n=0}^\infty$ (with j fixed) with general ω_m were first studied by the author in Sidi [22,24]. See also [6, Chapter 19] for additional developments.

References

[1] A. Sidi, Numerical quadrature and nonlinear sequence transformations; unified rules for efficient computation of integrals with algebraic and logarithmic endpoint singularities, Math. Comp. 35 (1980) 851–874.
[2] A. Sidi, Numerical quadrature rules for some infinite range integrals, Math. Comp. 38 (1982) 127–142.
[3] D. Shanks, Nonlinear transformations of divergent and slowly convergent sequences, J. Math. Phys. 34 (1955) 1–42.
[4] G.A. Baker Jr., Essentials of Padé Approximants, Academic Press, New York, 1975.
[5] G.A. Baker Jr., P.R. Graves-Morris, Padé Approximants, second ed., Cambridge University Press, Cambridge, 1996.
[6] A. Sidi, Practical Extrapolation Methods: Theory and Applications, in: Cambridge Monographs on Applied and Computational Mathematics, vol. 10, Cambridge University Press, Cambridge, 2003.
[7] P.J. Davis, P. Rabinowitz, Methods of Numerical Integration, second ed., Academic Press, New York, 1984.
[8] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, third ed., Springer-Verlag, New York, 2002.
[9] A. Ralston, P. Rabinowitz, A First Course in Numerical Analysis, second ed., McGraw-Hill, New York, 1978.
[10] D. Levin, Development of non-linear transformations for improving convergence of sequences, Int. J. Comput. Math. B3 (1973) 371–388.
[11] D.S. Lubinsky, A. Sidi, Positive interpolatory quadrature rules generated by some biorthogonal polynomials, Math. Comp. 79 (2010) 845–855.
[12] V.I. Krylov, Approximate Calculation of Integrals, Macmillan, New York, 1962, Translated by A.H. Stroud.
[13] A. Sidi, D.S. Lubinsky, Biorthogonal polynomials and numerical integration formulas for infinite intervals, JNAIAM J. Numer. Anal. Ind. Appl. Math. 2 (2007) 209–226.
[14] A. Sidi, D.S. Lubinsky, On the zeros of some polynomials that arise in numerical quadrature and convergence acceleration, SIAM J. Numer. Anal. 20 (1983) 400–405.
[15] A. Sidi, Problems 5–8, in: H. Brass, G. Hämmerlin (Eds.), Numerical Integration III, in: ISNM, vol. 85, Birkhäuser, Basel, 1988, pp. 321–325.
[16] D.S. Lubinsky, A. Sidi, Strong asymptotics for polynomials biorthogonal to powers of log x, Analysis 14 (1994) 341–379.
[17] D.S. Lubinsky, A. Sidi, Zero distribution of composite polynomials, and polynomials biorthogonal to exponentials, Constr. Approx. 28 (2008) 343–371.
[18] G. Szegő, Orthogonal Polynomials, in: American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, Providence, Rhode Island, 1939.
[19] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
[20] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 1999.
[21] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.
[22] A. Sidi, Convergence properties of some nonlinear sequence transformations, Math. Comp. 33 (1979) 315–326.
[23] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980, Forth printing 1983.
[24] A. Sidi, Analysis of convergence of the T-transformation for power series, Math. Comp. 35 (1980) 833–850.