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Analysis of errors in some recent numerical quadrature formulas for periodic singular and hypersingular integrals via regularization

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ABSTRACT

Recently, we derived some new numerical quadrature formulas of trapezoidal rule type for the singular integrals $I^{(1)}[u] = \int_a^b (\cot\frac{\pi(x-t)}{T})u(x)\,dx$ and $I^{(2)}[u] = \int_a^b (\csc^2\frac{\pi(x-t)}{T})u(x)\,dx$, with $b-a=T$ and $u(x)$ a T-periodic continuous function on $\mathbb{R}$. These integrals are not defined in the regular sense, but are defined in the sense of Cauchy Principal Value and Hadamard Finite Part, respectively. With $h=(b-a)/n$, $n=1,2,\ldots,$ the numerical quadrature formulas $Q_n^{(1)}[u]$ for $I^{(1)}[u]$ and $Q_n^{(2)}[u]$ for $I^{(2)}[u]$ are

$$Q_n^{(1)}[u] = h\sum_{j=1}^{n} f(t+jh-h/2), \qquad f(x) = \left(\cot\frac{\pi(x-t)}{T}\right)u(x),$$

and

$$Q_n^{(2)}[u] = h\sum_{j=1}^{n} f(t+jh-h/2) - T^2u(t)h^{-1}, \qquad f(x) = \left(\csc^2\frac{\pi(x-t)}{T}\right)u(x).$$

We provided a complete analysis of the errors in these formulas under the assumption that $u \in C^\infty(\mathbb{R})$ and is T-periodic. We actually showed that,

$$I^{(1)}[u] - Q_n^{(1)}[u] = O\left(n^{-\mu}\right) \quad \text{and}$$
$$I^{(2)}[u] - Q_n^{(2)}[u] = O\left(n^{-\mu}\right) \quad \text{as } n\to\infty,\ \forall\mu>0.$$

In this note, we analyze the errors in these formulas under the weaker assumption that $u \in C^s(\mathbb{R})$ for some finite integer s. By first regularizing these integrals, we prove that, if $u^{(s+1)}$ is piecewise continuous, then

$$I^{(1)}[u] - Q_n^{(1)}[u] = o\left(n^{-s-1/2}\right) \quad \text{as } n\to\infty, \text{ if } s\geqslant 1, \quad \text{and}$$
$$I^{(2)}[u] - Q_n^{(2)}[u] = o\left(n^{-s+1/2}\right) \quad \text{as } n\to\infty, \text{ if } s\geqslant 2.$$

We also extend these results by imposing different smoothness conditions on $u^{(s+1)}$. Finally, we append suitable numerical examples.

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1. Introduction and background

Let $u(x)$ be a T-periodic function that is sufficiently smooth on $\mathbb{R}$ and consider the singular integrals

$$I^{(1)}[u] = \int_a^b \left(\cot \frac{\pi(x-t)}{T}\right) u(x)\,dx, \quad a < t < b;\ b - a = T, \tag{1.1}$$

and

$$I^{(2)}[u] = \int_a^b \left(\csc^2 \frac{\pi(x-t)}{T}\right) u(x)\,dx, \quad a < t < b;\ b - a = T. \tag{1.2}$$

Of these, $I^{(1)}[u]$ is known also as the *circular Hilbert transform* and is defined in the sense of *Cauchy Principal Value (CPV)*, while $I^{(2)}[u]$ is a so-called *hypersingular integral* and is defined in the sense of *Hadamard Finite Part (HFP)*. Note that the integrands in both integrals are T-periodic with nonintegrable singularities at $x = t$ in the interval of integration (a, b); namely, the integrand of $I^{(1)}[u]$ has a singularity of the form $(x-t)^{-1}$, while $I^{(2)}[u]$ has a singularity of the form $(x-t)^{-2}$. For the properties of CPV and HFP integrals, see Davis and Rabinowitz [3], Evans [4], or Kythe and Schäferkotter [5], for example.[1]

In the recent papers Sidi and Israeli [14] and Sidi [12], we derived trapezoidal rule type approximations to the integrals $I^{(1)}[u]$ and $I^{(2)}[u]$, respectively.[2] With $h = T/n$, $n = 1, 2, \ldots,$ these approximations are

$$Q_n^{(1)}[u] = h \sum_{j=1}^{n} f(t + jh - h/2), \qquad f(x) = \left(\cot \frac{\pi(x-t)}{T}\right) u(x), \quad \text{for } I^{(1)}[u], \tag{1.3}$$

and

$$Q_n^{(2)}[u] = h \sum_{j=1}^{n} f(t + jh - h/2) - T^2 u(t) h^{-1}, \qquad f(x) = \left(\csc^2 \frac{\pi(x-t)}{T}\right) u(x), \quad \text{for } I^{(2)}[u]. \tag{1.4}$$

We also derived the following results concerning the errors in these approximations under the assumption that $u \in C^\infty(\mathbb{R})$:

$$Q_n^{(1)}[u] - I^{(1)}[u] = O\left(n^{-\mu}\right) \quad \text{as } n \to \infty,\ \forall \mu > 0, \tag{1.5}$$

and

$$Q_n^{(2)}[u] - I^{(2)}[u] = O\left(n^{-\mu}\right) \quad \text{as } n \to \infty,\ \forall \mu > 0. \tag{1.6}$$

As is done in [12], both of these results can be derived by using one of the author's generalizations of the classical Euler–Maclaurin expansion given in Sidi [10, Theorem 2.3].[3]

In this work, we analyze the errors in the formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ under the weaker assumption that $u(x)$ is

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[1] The usual notation for integrals defined in the sense of the Cauchy Principal Value (CPV) is $⨍_a^b f(x)\,dx$, while for those defined in the sense of Hadamard Finite Part (HFP) it is $⨎_a^b f(x)\,dx$. For simplicity, in this work, we use $\int_a^b f(x)\,dx$ to denote both, as in (1.1) and (1.2).

[2] Actually, in [12], we treated singular integrals of the very general forms

$$I[g] = \int_a^b g(x)\left(\log|x-t|\right)^p |x-t|^\beta, \quad \beta \text{ real},\ p = 0, 1,\ a < t < b,$$

and

$$I[g] = \int_a^b g(x)(x-t)^\beta, \quad \beta = -1, -2, \ldots,\ a < t < b,$$

$g(x)$ being allowed to have arbitrary algebraic endpoint singularities. Of course, $I^{(1)}[u]$ and $I^{(2)}[u]$ are special cases of these.

[3] The classical Euler–Maclaurin (E–M) expansion pertains to integrals $\int_a^b g(x)\,dx$ whose integrands are regular throughout the (closed) interval $[a, b]$, whereas the generalized E–M expansions of [10] treat the case in which $g(x)$ has arbitrary algebraic endpoint singularities. For the case in which $g(x)$ has arbitrary algebraic-logarithmic endpoint singularities, see Sidi [9] and [11]. In all three papers [9], [10], and [11], the integrals $\int_a^b g(x)\,dx$ can be convergent or divergent; in case of divergence, they are defined in the sense of HFP.

It was observed in [12] that the CPV integrals are also sums of HFP integrals, and this fact was used in the derivation of their associated asymptotic expansions.

differentiable only a finite number of times on $\mathbb{R}$. Specifically, we assume that

$$u \in C^s(\mathbb{R}) \quad \text{and } T\text{-periodic}, \qquad u^{(s+1)} \text{ piecewise continuous.}^4 \tag{1.7}$$

In this sense, this note complements [12].

The next two theorems are the main results of this work. It is easy to see that, when $u \in C^\infty(\mathbb{R})$, the results in (1.5) and (1.6) are implied by these theorems.

Theorem 1.1. *Let $u(x)$ be as in (1.7). Provided $s \geqslant 1$,*

$$Q_n^{(1)}[u] - I^{(1)}[u] = o\big(n^{-s-1/2}\big) \quad \text{as } n \to \infty. \tag{1.8}$$

Theorem 1.2. *Let $u(x)$ be as in (1.7). Provided $s \geqslant 2$,*

$$Q_n^{(2)}[u] - I^{(2)}[u] = o\big(n^{-s+1/2}\big) \quad \text{as } n \to \infty. \tag{1.9}$$

The proofs of these results are the subject of the next section. In these proofs, we first *regularize* the integral representations of $I^{(1)}[u]$ and $I^{(2)}[u]$; namely, we express the latter as integrals defined in the *regular* sense. We make use of some standard techniques of Fourier analysis, which can be found in Davis [2], for example. We also make use of Theorems 1.3 and 1.4 that are stated below and that pertain to the *exactness* of the quadrature formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$. These theorems were proved in [12] and seem to be of interest in themselves.

The use of the quadrature formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ under the conditions in (1.7) is illustrated with suitable numerical examples in Section 3. In Section 4, we extend the main results to functions $u(x)$ assuming different conditions on $u^{(s+1)}(x)$ than the one in (1.7).

Below and throughout the remainder of this work, we will sometimes let

$$e_m(x) = e^{\mathrm{i}2m\pi x/T}, \quad m \text{ integer}, \tag{1.10}$$

for simplicity of notation.

Theorem 1.3. *With $e_m(x)$ as in (1.10) and with the convention that $\operatorname{sgn}(0) = 0$ throughout, the following are true:*

1. *The integral $I^{(1)}[e_m] = \int_a^b (\cot \frac{\pi(x-t)}{T})\, e_m(x)\,dx$, $b - a = T$, satisfies*

$$I^{(1)}[e_m] = \mathrm{i}T \operatorname{sgn}(m)\, e_m(t), \quad m = 0, \pm 1, \pm 2, \ldots. \tag{1.11}$$

2. *The quadrature formula $Q_n^{(1)}[e_m]$ for $I^{(1)}[e_m]$ satisfies*

$$Q_n^{(1)}[e_m] = I^{(1)}[e_m] = \mathrm{i}T \operatorname{sgn}(m)\, e_m(t), \quad m = 0, \pm 1, \ldots, \pm(n-1), \tag{1.12}$$

while for arbitrary m, we have

$$Q_n^{(1)}[e_m] = \mathrm{i}T \operatorname{sgn}(m) V_{m,n}\, e_m(t), \tag{1.13}$$

with $V_{-m,n} = V_{m,n}$, and

$$V_{m,n} = \begin{cases} 0 & \text{if } m = 0, \\ 1 & \text{if } |m| = 1, \ldots, n-1, \end{cases} \quad \text{and} \quad V_{\pm m,n} = V_{kn+r,n} = (-1)^k V_{r,n}, \tag{1.14}$$

where k and r are unique integers, $k \geqslant 0$ and $0 \leqslant r \leqslant n-1$, such that $|m| = kn + r$, in which case,

$$Q_n^{(1)}[e_m] - I^{(1)}[e_m] = \mathrm{i}T \operatorname{sign}(m)(V_{m,n} - 1)\, e_m(t). \tag{1.15}$$

3. *If $u(x)$ is of the form*

$$u(x) = \sum_{m=-(n-1)}^{n-1} c_m e_m(x), \tag{1.16}$$

then the quadrature formula $Q_n^{(1)}[u]$ for $I^{(1)}[u]$, $b - a = T$, satisfies

$$Q_n^{(1)}[u] = I^{(1)}[u]. \tag{1.17}$$

[4] A function $g(x)$ is said to be piecewise continuous on $[a, b]$ if it is continuous everywhere in $[a, b]$ except at finitely many points, $y_1, \ldots, y_r$ say, at which it has finite right-hand and left-hand limits $g(y_k+)$ and $g(y_k-)$.

Note that part 1 of Theorem 1.3 can already be found in Lifanov [6].

Theorem 1.4. *With $e_m(x)$ as in (1.10), the following are true:*

1. *The integral $I^{(2)}[e_m] = \int_a^b (\csc^2 \frac{\pi(x-t)}{T})\, e_m(x)\,dx$, $b-a=T$, satisfies*
$$I^{(2)}[e_m] = -2T|m|e_m(t), \quad m = 0, \pm 1, \pm 2, \ldots. \tag{1.18}$$
2. *The quadrature formula $Q_n^{(2)}[e_m]$ for $I^{(2)}[e_m]$ satisfies*
$$Q_n^{(2)}[e_m] = I_n^{(2)}[e_m] = -2T|m|e_m(t), \quad m = 0, \pm 1, \pm 2, \ldots, \pm n, \tag{1.19}$$
while for arbitrary m, we have
$$Q_n^{(2)}[e_m] = T\big[(-1)^k(n-2r) - n\big]\, e_m(t), \tag{1.20}$$
where k and r are unique integers, $k \geqslant 0$ and $0 \leqslant r \leqslant n-1$, such that $|m| = kn + r$, in which case,
$$Q_n^{(2)}[e_m] - I^{(2)}[e_m] = T\big\{\big[(-1)^k - 1\big](n-2r) + 2kn\big\} e_m(t). \tag{1.21}$$
3. *If $u(x)$ is of the form*
$$u(x) = \sum_{m=-n}^{n} c_m e_m(x), \tag{1.22}$$
then the quadrature formula $Q_n^{(2)}[u]$ for $I^{(2)}[u]$, $b-a=T$, satisfies
$$Q_n^{(2)}[u] = I^{(2)}[u]. \tag{1.23}$$

Note that part 1 of Theorem 1.4 can already be found in Lifanov and Poltavskii [7].

Before closing, we would like to mention a recent development pertaining to the quadrature formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$. We observe that these formulas make use of the *integrand* values $f(t+jh-h/2)$. Because the functions $f(x)$ have *nonintegrable* singularities at $x=t$, $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ are prone to roundoff errors when computed in floating-point arithmetic, and these errors increase without bound as $h \to 0$ (equivalently, as $n \to \infty$). Indeed, even though the theoretical errors $Q_n^{(1)}[u] - I^{(1)}[u]$ and $Q_n^{(2)}[u] - I^{(2)}[u]$ tend to zero as $n \to \infty$ by Theorems 1.1 and 1.2, the actual errors in the computed $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ first decrease (in accordance with Theorems 1.1 and 1.2) as n increases up to a certain value, say n_0, and then start increasing, thus diminishing the best accuracy achieved for $n = n_0$ gradually until no accuracy is left. The issue of roundoff errors has been analyzed in detail in Sidi [13, Section 7] recently, the main conclusion being that roundoff errors in computed $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ increase mildly, like $\log n$ and n, respectively. Thus, using a floating-point arithmetic of relatively high precision, sufficiently high accuracy can be achieved for $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ before the roundoff errors and the theoretical errors $Q_n^{(1)}[u] - I^{(1)}[u]$ and $Q_n^{(2)}[u] - I^{(2)}[u]$ become of comparable size.

2. Regularization of $I^{(1)}[u]$ and $I^{(2)}[u]$ and proofs of main results

2.1. Regularization of $I^{(1)}[u]$ and $I^{(2)}[u]$

To prove (1.8) and (1.9), we cannot make direct use of the representations of $I^{(1)}[u]$ and $I^{(2)}[u]$ given in (1.1) and (1.2) as the integrands in these representations are not integrable in the regular sense. Therefore, we look for integral representations that are defined in the regular sense. Fortunately, such *regularized* representations can be derived in a straightforward way, and we provide these derivations in Theorems 2.1 and 2.2 next. We believe that, despite their simplicity, these regularized forms are of interest by themselves.

Theorem 2.1. *Let $u \in C(\mathbb{R})$ be T-periodic, and let u' be piecewise continuous. Then*
$$I^{(1)}[u] = -\frac{T}{\pi}\int_a^b \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right) u'(x)\,dx. \tag{2.1}$$

Proof. Making the variable transformation $x = t + \frac{T}{2\pi}y$ in (1.1), and invoking the assumption that the integrand is T-periodic, we obtain
$$I^{(1)}[u] = \frac{T}{2\pi}\int_{-\pi}^{\pi}\left(\cot\frac{y}{2}\right) v(y)\,dy, \quad v(y) = u\left(t + \frac{T}{2\pi}y\right). \tag{2.2}$$

Clearly, $v \in C(\mathbb{R})$ and is 2π-periodic, and v' is piecewise continuous. In addition, $(\cot \frac{y}{2})v(y)$, the integrand of (2.2), has a singularity of the type y^{-1} at $y = 0$ in the interval $(-\pi, \pi)$.

We next recall the known definition of the CPV integral, namely,

$$\int_a^b \frac{g(x)}{x-t}\,dx = \lim_{\epsilon\to 0+}\left[\left(\int_a^{t-\epsilon} + \int_{t+\epsilon}^b\right)\frac{g(x)}{x-t}\,dx\right], \quad a < t < b,$$

which, in our case, becomes

$$\int_{-\pi}^{\pi}\left(\cot\frac{y}{2}\right)v(y)\,dy = \lim_{\epsilon\to 0+}\left[\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right)\left(\cot\frac{y}{2}\right)v(y)\,dy\right]. \tag{2.3}$$

Now, upon integrating $\int(\cot\frac{y}{2})v(y)\,dy$ by parts, we have

$$\int\left(\cot\frac{y}{2}\right)v(y)\,dy = 2\left(\log\left|\sin\frac{y}{2}\right|\right)v(y) - 2\int\left(\log\left|\sin\frac{y}{2}\right|\right)v'(y)\,dy.$$

[Observe that the integral $\int \log(|\sin\frac{y}{2}|)v'(y)\,dy$ is regular through $y = 0$.] Substituting this on the right-hand side of (2.3), and noting that $\log|\sin\frac{\pm\pi}{2}| = 0$ and

$$\begin{aligned}\left(\log\left|\sin\frac{\pm\epsilon}{2}\right|\right)v(\pm\epsilon) &= \left[\left(\log\frac{\epsilon}{2}\right) + O\left(\epsilon^2\right)\right]\left[v(0) + O(\epsilon)\right] \quad \text{as } \epsilon \to 0+\\ &= v(0)\left(\log\frac{\epsilon}{2}\right) + O(\epsilon\log\epsilon) \quad \text{as } \epsilon\to 0+,\end{aligned}$$

we see that the sum of the contributions of the integrated term $2\log(|\sin\frac{y}{2}|)v(y)$ from the endpoints $\pm\pi$ and $\pm\epsilon$ tends to zero as $\epsilon \to 0+$. We thus obtain

$$\int_{-\pi}^{\pi}\left(\cot\frac{y}{2}\right)v(y)\,dy = -2\int_{-\pi}^{\pi}\left(\log\left|\sin\frac{y}{2}\right|\right)v'(y)\,dy.$$

Substituting this in (2.2) and invoking $v'(y) = \frac{T}{2\pi}u'(t + \frac{T}{2\pi}y)$ and going back to the variable x, we obtain the result in (2.1). □

Theorem 2.2. *Let $u \in C^1(\mathbb{R})$ be T-periodic, and let u'' be piecewise continuous. Then*

$$I^{(2)}[u] = -\left(\frac{T}{\pi}\right)^2\int_a^b\left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right)u''(x)\,dx. \tag{2.4}$$

Proof. Making the variable transformation $x = t + \frac{T}{2\pi}y$ in (1.2), and invoking the assumption that the integrand is T-periodic, we obtain

$$I^{(2)}[u] = \frac{T}{2\pi}\int_{-\pi}^{\pi}\left(\csc^2\frac{y}{2}\right)v(y)\,dy, \quad v(y) = u\left(t + \frac{T}{2\pi}y\right). \tag{2.5}$$

Clearly, $v \in C^1(\mathbb{R})$ and is 2π-periodic, and v'' is piecewise continuous. In addition, $(\csc^2\frac{y}{2})v(y)$, the integrand of (2.2), has a singularity of the type y^{-2} at $y = 0$ in the interval $(-\pi, \pi)$.

We next recall the known definition of the hypersingular integral, namely,

$$\int_a^b \frac{g(x)}{(x-t)^2}\,dx = \lim_{\epsilon\to 0+}\left[\left(\int_a^{t-\epsilon} + \int_{t+\epsilon}^b\right)\frac{g(x)}{(x-t)^2}\,dx - \frac{2g(t)}{\epsilon}\right], \quad a < t < b,$$

which, in our case, becomes

$$\int_{-\pi}^{\pi}\left(\csc^2\frac{y}{2}\right)v(y)\,dy = \lim_{\epsilon\to 0+}\left[\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right)\left(\csc^2\frac{y}{2}\right)v(y)\,dy - \frac{8v(0)}{\epsilon}\right]. \tag{2.6}$$

Now, upon integrating $\int(\csc^2\frac{y}{2})v(y)\,dy$ by parts, we have

$$\int\left(\csc^2\frac{y}{2}\right)v(y)\,dy=-2\left(\cot\frac{y}{2}\right)v(y)+2\int\left(\cot\frac{y}{2}\right)v'(y)\,dy.$$

Substituting this in (2.6), and noting that $\cot\frac{\pm\pi}{2}=0$ and

$$\begin{aligned}\left(\cot\frac{\pm\epsilon}{2}\right)v(\pm\epsilon)&=\frac{2}{\pm\epsilon}\left[1+O\left(\epsilon^2\right)\right]\left[v(0)\pm v'(0)\epsilon+O\left(\epsilon^2\right)\right]\quad\text{as }\epsilon\to0+\\&=\frac{2}{\pm\epsilon}\left[v(0)\pm v'(0)\epsilon+O\left(\epsilon^2\right)\right]\quad\text{as }\epsilon\to0+,\end{aligned}$$

we see that the sum of the contributions of the integrated term $-2(\cot\frac{y}{2})v(y)$ from the endpoints $\pm\pi$ and $\pm\epsilon$ with the term $-\frac{8v(0)}{\epsilon}$ tends to zero as $\epsilon\to0+$. We thus obtain

$$\begin{aligned}\int_{-\pi}^{\pi}\left(\csc^2\frac{y}{2}\right)v(y)\,dy&=2\lim_{\epsilon\to0+}\left[\left(\int_{-\pi}^{-\epsilon}+\int_{\epsilon}^{\pi}\right)\left(\cot\frac{y}{2}\right)v'(y)\,dy\right]\\&=2\int_{-\pi}^{\pi}\left(\cot\frac{y}{2}\right)v'(y)\,dy,\end{aligned}$$

the second equality following from (2.3). [Note that (2.3) is applicable to our case since v' and v'' satisfy the conditions of Theorem 2.1.] We now apply Theorem 2.1 to the integral $\int_{-\pi}^{\pi}(\cot\frac{y}{2})v'(y)\,dy$, and obtain

$$\int_{-\pi}^{\pi}\left(\csc^2\frac{y}{2}\right)v(y)\,dy=2\int_{-\pi}^{\pi}\left(\cot\frac{y}{2}\right)v'(y)\,dy=-4\int_{-\pi}^{\pi}\left(\log\left|\sin\frac{y}{2}\right|\right)v''(y)\,dy.$$

Substituting this in (2.5) and invoking $v''(y)=(\frac{T}{2\pi})^2u''(t+\frac{T}{2\pi}y)$ and going back to the variable x, we obtain the result in (2.4). $\square$

Note that the kernel $(\log|\sin\frac{\pi(x-t)}{T}|)$ that is present in both (2.1) and (2.4) is negative and integrable in the regular sense since its only singularity at $x=t$ is logarithmic.

Note also that we can unify the results in (2.1) and (2.4) and write

$$I^{(q)}[u]=-\left(\frac{T}{\pi}\right)^q\int_a^b\left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right)u^{(q)}(x)\,dx,\quad q=1,2,$$
$$\text{provided } u\in C^{q-1}(\mathbb{R})\text{ and }T\text{-periodic},\ u^{(q)}\text{ piecewise continuous.}\tag{2.7}$$

This also enables us to unify their proofs, as we will see later.

2.2. Proofs of main results

We now turn to the proofs of Theorems 1.1 and 1.2. We start with the following simple lemma concerning infinite sequences in the Hilbert space l^2:

Lemma 2.3. *Let*

$$d_m=\frac{\rho_m}{m^p},\quad m=1,2,\ldots;\qquad\{\rho_m\}_{m=1}^{\infty}\in l^2.$$

Then,

$$\sum_{m=N}^{\infty}d_m=o\left(N^{-p+1/2}\right)\quad\textit{as } N\to\infty,\ \textit{provided } p>1/2.$$

Proof. First, $\sum_{m=1}^{\infty}|\rho_m|^2<\infty$. Next, it is easy to see that the sequence $\{m^{-p}\}_{m=1}^{\infty}$ belongs to l^2, just as $\{\rho_m\}_{m=1}^{\infty}$. Therefore, by the Cauchy–Schwartz inequality,

$$\left|\sum_{m=N}^{\infty} d_m\right| = \left|\sum_{m=N}^{\infty} \frac{\rho_m}{m^p}\right| \leqslant \left(\sum_{m=N}^{\infty} \frac{1}{m^{2p}}\right)^{1/2} \left(\sum_{m=N}^{\infty} |\rho_m|^2\right)^{1/2}.$$

The result now follows by invoking

$$\sum_{m=N}^{\infty} |\rho_m|^2 = o(1) \quad \text{as } N\to\infty \quad \text{and} \quad \sum_{m=N}^{\infty} m^{-r} = O\left(N^{-r+1}\right) \quad \text{as } N\to\infty,\ r>1. \qquad \square$$

For simplicity, we let $T=2\pi$ and $[a,b]=[-\pi,\pi]$ in the sequel. Let the Fourier series of $u(x)$ be as in

$$u(x) = \sum_{m=-\infty}^{\infty} c_m e^{\mathrm{i}mx}; \quad c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}mx} u(x)\,dx,\ m=0,\pm1,\pm2,\ldots. \tag{2.8}$$

By (1.7), $u\in C^s(\mathbb{R})$, $u^{(s+1)}$ being piecewise continuous, we have by repeated integration by parts

$$c_m = \frac{1}{2\pi}\left[(-1)^m \sum_{k=0}^{s} \frac{u^{(k)}(-\pi)-u^{(k)}(\pi)}{(\mathrm{i}m)^{k+1}} + \frac{1}{(\mathrm{i}m)^{s+1}}\int_{-\pi}^{\pi} e^{-\mathrm{i}mx} u^{(s+1)}(x)\,dx\right].$$

By 2π-periodicity of $u^{(k)}(x)$ for $k=0,1,\ldots,s$, this becomes

$$c_m = \frac{\beta_m}{(\mathrm{i}m)^{s+1}}, \quad \beta_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}mx} u^{(s+1)}(x)\,dx. \tag{2.9}$$

Then the Fourier series of $u'(x)$ (when $s\geqslant 1$) and of $u''(x)$ (when $s\geqslant 2$) over $[-\pi,\pi]$ converge absolutely and uniformly everywhere,[5] and are given as in

$$u'(x) = u^{(1)}(x) = \sum_{m=-\infty}^{\infty} c_m^{(1)} e^{\mathrm{i}mx}; \quad c_m^{(1)} = \mathrm{i}m c_m = \frac{\beta_m}{(\mathrm{i}m)^s}, \tag{2.10}$$

and

$$u''(x) = u^{(2)}(x) = \sum_{m=-\infty}^{\infty} c_m^{(2)} e^{\mathrm{i}mx}; \quad c_m^{(2)} = (\mathrm{i}m)^2 c_m = \frac{\beta_m}{(\mathrm{i}m)^{s-1}}. \tag{2.11}$$

For simplicity of notation, let us express (2.7) as in

$$I^{(q)}[u] = \int_{-\pi}^{\pi} \phi_q(x-t) u^{(q)}(x)\,dx, \quad \phi_q(y) = -2^q\left(\log\left|\sin\frac{y}{2}\right|\right). \tag{2.12}$$

Then, with $e_m(x)=e^{\mathrm{i}mx}$, as in (1.10), we have

$$\sum_{m=-N}^{N} c_m I^{(q)}[e_m] = I^{(q)}\left[\sum_{m=-N}^{N} c_m e_m\right] = I^{(q)}\left[u - \sum_{|m|>N} c_m e_m\right],$$

which, by (2.10), (2.11), and (2.12), becomes

$$\begin{aligned}\sum_{m=-N}^{N} c_m I^{(q)}[e_m] &= I^{(q)}[u] - I^{(q)}\left[\sum_{|m|>N} c_m e_m\right]\\ &= I^{(q)}[u] - \int_{-\pi}^{\pi} \phi_q(x-t)\left[\sum_{|m|>N} c_m^{(q)} e_m(x)\right] dx.\end{aligned} \tag{2.13}$$

[5] It is known that if $w\in C(\mathbb{R})$ and is 2π-periodic, w' being piecewise continuous, then the Fourier series of $w(x)$ on $[-\pi,\pi]$ converges to $w(x)$ absolutely and uniformly on $\mathbb{R}$.

Now, by the fact that $\phi(x-t)$ is in $L^1[-\pi,\pi]$, and the series $\sum_{|m|>N} c_m^{(q)} e_m(x)$, both for $q=1$ and $q=2$, converge absolutely and uniformly everywhere, we have

$$\left|\int_{-\pi}^{\pi} \phi_q(x-t)\left[\sum_{|m|>N} c_m^{(q)} e_m(x)\right] dx\right| \leqslant \left[\int_{-\pi}^{\pi} \left|\phi_q(x-t)\right| dx\right]\left(\max_{|x|\leqslant\pi}\left|\sum_{|m|>N} c_m^{(q)} e_m(x)\right|\right)$$
$$\leqslant \left[\int_{-\pi}^{\pi} \left|\phi_q(x)\right| dx\right] \sum_{|m|>N} \left|c_m^{(q)}\right|, \tag{2.14}$$

since $\phi_q(y)$ is 2π-periodic and hence $\int_{-\pi}^{\pi} |\phi_q(x-t)|\,dx = \int_{-\pi}^{\pi} |\phi_q(x)|\,dx$, independently of t. By our assumptions on s, it follows that $\lim_{N\to\infty} \sum_{|m|>N} |c_m^{(q)}| = 0$ by Lemma 2.3. Consequently,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \phi_q(x-t)\left[\sum_{|m|>N} c_m^{(q)} e_m(x)\right] dx = 0,$$

and hence

$$\lim_{N\to\infty} \sum_{m=-N}^{N} c_m I^{(q)}[e_m] = I^{(q)}[u].$$

Thus, we have shown that

$$I^{(q)}[u] = \sum_{m=-\infty}^{\infty} c_m I^{(q)}[e_m], \quad q=1,2. \tag{2.15}$$

As for $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$, we note that they are both *finite* linear combinations of values of $u(x)$ as can be seen from (1.3) and (1.4), respectively. Therefore, we have immediately

$$Q_n^{(q)}[u] = Q_n^{(q)}\left[\sum_{m=-\infty}^{\infty} c_m e_m\right] = \sum_{m=-\infty}^{\infty} c_m Q^{(q)}[e_m], \quad q=1,2. \tag{2.16}$$

Combining (2.16) with (2.15), we obtain

$$Q_n^{(q)}[u] - I^{(q)}[u] = \sum_{m=-\infty}^{\infty} c_m\left(Q_n^{(q)}[e_m] - I^{(q)}[e_m]\right), \quad q=1,2. \tag{2.17}$$

Upon invoking part 3 of Theorem 1.3 and part 3 of Theorem 1.4 in (2.17), we first obtain

$$Q_n^{(q)}[u] - I^{(q)}[u] = \sum_{|m|>N_q} c_m\left(Q_n^{(q)}[e_m] - I^{(q)}[e_m]\right), \quad q=1,2, \tag{2.18}$$

where we have set

$$N_1 = n-1, \qquad N_2 = n. \tag{2.19}$$

We next take moduli on both sides, and by manipulating (1.15) and (1.21), respectively, we obtain

$$\left|Q_n^{(1)}[u] - I^{(1)}[u]\right| \leqslant 4\pi \sum_{m=n}^{\infty}\left(|c_m| + |c_{-m}|\right) \tag{2.20}$$

and

$$\left|Q_n^{(2)}[u] - I^{(2)}[u]\right| \leqslant 8\pi \sum_{m=n+1}^{\infty} m\left(|c_m| + |c_{-m}|\right). \tag{2.21}$$

Finally, we substitute (2.9) in (2.20) and (2.21) and apply Lemma 2.3 with $\rho_m = \beta_{\pm m}$, and with $p = s+1$ for (2.20) and $p = s$ in (2.21). The results in (1.8) and (1.9) now follow.

Note. Observe that what we have in (2.18) are actually closed-form error formulas for $Q_n^{(q)}[u] - I^{(q)}[u]$, once we recall the explicit expressions for $Q_n^{(q)}[e_m] - I^{(q)}[e_m]$ given in (1.15) and (1.21).

3. Numerical examples

To illustrate our results, we have applied our quadrature formulas to the cases in which $u(x) = \overline{B}_p(x)$, where $\overline{B}_p(x)$ is the pth *periodic Bernoulli function*. $\overline{B}_p(x)$ is defined as the 1-periodic extension of $B_p(x)$, the *Bernoulli polynomial* of degree p, namely,

$$\overline{B}_p(x) = B_p(x-k), \quad \text{for } x \in [k, k+1),\ k = 0, \pm 1, \pm 2, \ldots.$$

For Bernoulli polynomials and their periodic extensions, see Abramowitz and Stegun [1, Chapter 23] or Sidi [8, Appendix D], for example.

Now, the Bernoulli polynomials satisfy

$$B_p^{(k)}(0) = B_p^{(k)}(1), \quad k = 0, 1, \ldots, p-2, \quad \text{but} \quad B_p^{(p-1)}(0) \neq B_p^{(p-1)}(1), \quad p \geqslant 2. \tag{3.1}$$

Consequently,

$$\overline{B}_p \in C^{p-2}(\mathbb{R}) \quad \text{and} \quad \overline{B}_p^{(p-1)} \quad \text{piecewise continuous on } \mathbb{R},\ p = 2, 3, \ldots.$$

For brevity, from here on, we concentrate on $\overline{B}_p(x)$ with $p = 2r$, $r = 1, 2, \ldots$. From [1, p. 805, Eq. 23.1.18], $B_{2r}(x)$ has the Fourier series

$$B_{2r}(x) = (-1)^{r-1} \frac{2\,(2r)!}{(2\pi)^{2r}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2r}}, \quad x \in [0, 1],\ r = 1, 2, \ldots.$$

Let $e_k(x) = \exp(\mathrm{i}2k\pi x)$ as in (1.10). Applying parts 1 of Theorems 1.3 and 1.4 to $w_k(x) = \cos 2k\pi x = \frac{1}{2}[e_k(x) + e_{-k}(x)]$, with $[a, b] = [0, 1]$ and $T = 1$, we obtain, respectively,

$$I^{(1)}[w_k] = -\sin 2k\pi t \quad \text{and} \quad I^{(2)}[w_k] = -2k \cos 2k\pi t, \quad k = 1, 2, \ldots.$$

Thus, by (2.15), we have the following series representations for $I^{(1)}[\overline{B}_{2r}]$ and $I^{(2)}[\overline{B}_{2r}]$:

$$I^{(1)}[\overline{B}_{2r}] = \int_0^1 \bigl[\cot \pi(x-t)\bigr] \overline{B}_{2r}(x)\,dx = (-1)^r \frac{2\,(2r)!}{(2\pi)^{2r}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi t}{k^{2r}}, \quad r = 1, 2, \ldots,$$

$$I^{(2)}[\overline{B}_{2r}] = \int_0^1 \bigl[\csc^2 \pi(x-t)\bigr] \overline{B}_{2r}(x)\,dx = (-1)^r \frac{4\,(2r)!}{(2\pi)^{2r}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi t}{k^{2r-1}}, \quad r = 1, 2, \ldots.$$

Here, we present the results of the computations we have carried out in quadruple-precision floating-point arithmetic by applying the quadrature formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ with $u(x) = \overline{B}_6(x)$ (i.e., $r = 3$ above), and $t = 0.3$. Note that

$$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.$$

Consequently,

$$I^{(1)}[\overline{B}_6] = -\frac{45}{2\pi^6} \sum_{k=1}^{\infty} \frac{\sin 2k\pi t}{k^6} \quad \text{and} \quad I^{(2)}[\overline{B}_6] = -\frac{45}{\pi^6} \sum_{k=1}^{\infty} \frac{\cos 2k\pi t}{k^5}.$$

Since these series converge relatively quickly, they can be used to obtain the exact values of $I^{(1)}[u]$ and $I^{(2)}[u]$ conveniently; for $t = 0.3$, these are

$$I^{(1)}[u] = -0.02202945729223988\ldots \quad \text{and} \quad I^{(2)}[u] = 0.01548960081292585\ldots.$$

Table 1 contains the relative errors

$$\mathcal{E}_n^{(1)} = \left| \frac{Q_n^{(1)}[u] - I^{(1)}[u]}{I^{(1)}[u]} \right| \quad \text{and} \quad \mathcal{E}_n^{(2)} = \left| \frac{Q_n^{(2)}[u] - I^{(2)}[u]}{I^{(2)}[u]} \right|$$

in the results obtained by applying the quadrature formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ with $t = 0.3$ and $n = 2^k$, $1 \leqslant k \leqslant 14$. Note that $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$ achieve accuracies of eight significant decimal digits already for $n = 16$. The last two numbers for $\mathcal{E}_n^{(2)}$ at the bottom of the table exhibit the effect of the roundoff errors we alluded to in the last paragraph of Section 1.

Table 1
Quadruple-precision floating-point numerical results for the integrals $I^{(1)}[u]$ and $I^{(2)}[u]$ in Section 3, with $[a,b]=[0,1]$, $T=1$, $u(x)=\overline{B}_6(x)$ and $t=0.3$ throughout. Here $\mathcal{E}_n^{(1)}=|Q_n^{(1)}[u]-I^{(1)}[u]|/|I^{(1)}[u]|$ and $\mathcal{E}_n^{(2)}=|Q_n^{(2)}[u]-I^{(2)}[u]|/|I^{(2)}[u]|$. Of course, $h=1/n$ in the quadrature formulas $Q_n^{(1)}[u]$ and $Q_n^{(2)}[u]$.

n	$\mathcal{E}_n^{(1)}$	$\mathcal{E}_n^{(2)}$
2	1.12E−02	6.98E−03
4	2.16E−04	2.53E−04
8	7.45E−07	3.20E−06
16	2.51E−09	8.64E−09
32	6.59E−10	1.87E−09
64	4.63E−12	1.28E−11
128	1.37E−13	4.15E−13
256	8.15E−16	2.50E−15
512	3.52E−17	1.05E−16
1024	2.18E−19	6.50E−19
2048	8.49E−21	2.56E−20
4096	5.21E−23	1.50E−24
8192	2.08E−24	1.61E−22
16 384	6.77E−27	1.55E−22

4. Concluding remarks

The functions $u(x)$ that we treated above were assumed to be such that $u^{(s+1)}$ is piecewise continuous. This lead to the useful conclusion that $u^{(s+1)}\in L^2[a,b]$, hence the sequence $\{\beta_m\}_{m=-\infty}^{\infty}$ is in the space l^2, which allowed the asymptotic results of Theorems 1.1 and 1.2 to hold. Of course, the *same* results are obtained if we allow $u^{(s+1)}$ to satisfy the weaker condition that $u^{(s+1)}\in L^2[a,b]$.

We can also assume that $u(x)$ is in the Hölder class $C^{s+1,\alpha}[a,b]$, $0<\alpha\leqslant 1$, that is, $u\in C^{s+1}[a,b]$ such that $u^{(i)}(x)$, $i=0,1,\dots,s$, are all T-periodic, $T=b-a$ and $u^{(s+1)}$ is in the Hölder class $C^{0,\alpha}[a,b]$.[6] This imposes a stronger condition on $u^{(s+1)}$; namely, $u^{(s+1)}$ is now continuous on $[a,b]$. Clearly, Theorems 1.1 and 1.2 apply to this case with no changes since $u^{(s+1)}$ is continuous on $\mathbb{R}$. For $1/2<\alpha\leqslant 1$, *better* results can be obtained, however. First, it is known that, in (2.9), $\beta_m=O(m^{-\alpha})$ as $m\to\infty$, so that $c_m=O(m^{-s-\alpha-1})$ as $|m|\to\infty$. Thus, provided $s>1-\alpha$, the Fourier series of $u'(x)$ converges absolutely and uniformly, hence (2.20) holds, from which we obtain

$$Q_n^{(1)}[u]-I^{(1)}[u]=o\left(n^{-s-\alpha}\right)\quad \text{as } n\to\infty,\ \text{if } s>1-\alpha. \tag{4.2}$$

Similarly, provided $s>2-\alpha$, the Fourier series of $u''(x)$ converges absolutely and uniformly, and hence, (2.21) holds, from which we obtain

$$Q_n^{(2)}[u]-I^{(2)}[u]=o\left(n^{-s-\alpha+1}\right)\quad \text{as } n\to\infty,\ \text{if } s>2-\alpha. \tag{4.3}$$

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[6] A function $g(x)$ is in $C^{0,\alpha}[a,b]$, $0<\alpha\leqslant 1$, if $|g(x)-g(y)|\leqslant L|x-y|^{\alpha}$, for all $x,y\in[a,b]$, for some constant $L>0$.