Numer Algor (2017) 74:937–949
DOI 10.1007/s11075-016-0178-8

ORIGINAL PAPER

On a vectorized version of a generalized Richardson extrapolation process

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Received: 10 May 2016 / Accepted: 12 July 2016 / Published online: 5 August 2016

Abstract Let $\{\boldsymbol{x}_m\}$ be a vector sequence that satisfies

$$\boldsymbol{x}_m \sim \boldsymbol{s} + \sum_{i=1}^{\infty} \alpha_i \boldsymbol{g}_i(m) \quad \text{as } m \to \infty,$$

$\boldsymbol{s}$ being the limit or antilimit of $\{\boldsymbol{x}_m\}$ and $\{\boldsymbol{g}_i(m)\}_{i=1}^{\infty}$ being an asymptotic scale as $m \to \infty$, in the sense that

$$\lim_{m\to\infty} \frac{\|\boldsymbol{g}_{i+1}(m)\|}{\|\boldsymbol{g}_i(m)\|} = 0, \quad i = 1, 2, \ldots .$$

The vector sequences $\{\boldsymbol{g}_i(m)\}_{m=0}^{\infty}$, $i = 1, 2, \ldots,$ are known, as well as $\{\boldsymbol{x}_m\}$. In this work, we analyze the convergence and convergence acceleration properties of a vectorized version of the generalized Richardson extrapolation process that is defined via the equations

$$\sum_{i=1}^{k} \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle \widetilde{\alpha}_i = \langle \boldsymbol{y}, \Delta \boldsymbol{x}_m\rangle, \quad n \le m \le n+k-1; \quad \boldsymbol{s}_{n,k} = \boldsymbol{x}_n + \sum_{i=1}^{k} \widetilde{\alpha}_i \boldsymbol{g}_i(n),$$

$\boldsymbol{s}_{n,k}$ being the approximation to $\boldsymbol{s}$. Here, $\boldsymbol{y}$ is some nonzero vector, $\langle\cdot\,,\cdot\rangle$ is an inner product, such that $\langle \alpha\boldsymbol{a}, \beta\boldsymbol{b}\rangle = \overline{\alpha}\beta\langle \boldsymbol{a}, \boldsymbol{b}\rangle$, and $\Delta\boldsymbol{x}_m = \boldsymbol{x}_{m+1} - \boldsymbol{x}_m$ and $\Delta\boldsymbol{g}_i(m) = \boldsymbol{g}_i(m+1) - \boldsymbol{g}_i(m)$. By imposing a minimal number of reasonable additional conditions on the $\boldsymbol{g}_i(m)$, we show that the error $\boldsymbol{s}_{n,k} - \boldsymbol{s}$ has a full asymptotic

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expansion as $n \to \infty$. We also show that actual convergence acceleration takes place, and we provide a complete classification of it.

Keywords Acceleration of convergence · Vector extrapolation methods · Vectorized generalized Richardson extrapolation process

Mathematics Subject Classification (2010) 65B05 · 65B10 · 40A05 · 40A25

1 Introduction

Let $\mathbb{X}$ be a finite or infinite dimensional linear inner product space with the inner product $\langle\cdot\,,\cdot\rangle$ defined such that $\langle\alpha\boldsymbol{a}, \beta\boldsymbol{b}\rangle = \overline{\alpha}\beta\langle\boldsymbol{a}, \boldsymbol{b}\rangle$, and let $\|\cdot\|$ be the norm induced by this inner product, namely, $\|z\| = \sqrt{\langle z, z\rangle}$.

Let $\{\boldsymbol{x}_m\}$ be a vector sequence in $\mathbb{X}$, and let $\boldsymbol{x}_m$ have an asymptotic expansion of the form

$$\boldsymbol{x}_m \sim \boldsymbol{s} + \sum_{i=1}^{\infty} \alpha_i \boldsymbol{g}_i(m) \quad \text{as } m \to \infty, \tag{1.1}$$

$\boldsymbol{s}$ being the limit or antilimit of $\{\boldsymbol{x}_m\}$ and $\{\boldsymbol{g}_i(m)\}_{i=1}^{\infty}$ being an asymptotic scale as $m \to \infty$, in the sense that

$$\lim_{m\to\infty} \frac{\|\boldsymbol{g}_{i+1}(m)\|}{\|\boldsymbol{g}_i(m)\|} = 0, \quad i = 1, 2, \ldots\,. \tag{1.2}$$

The vector sequences $\{\boldsymbol{g}_i(m)\}_{m=0}^{\infty}$, $i = 1, 2, \ldots,$ are known, as well as $\{\boldsymbol{x}_m\}$. The scalars α_i do not have to be known. By (1.1), we mean

$$\left\| \boldsymbol{x}_m - \boldsymbol{s} - \sum_{i=1}^{r} \alpha_i \boldsymbol{g}_i(m) \right\| = o(\|\boldsymbol{g}_r(m)\|) \quad \text{as } m \to \infty, \quad \forall\, r \geq 1. \tag{1.3}$$

Of course, the summation $\sum_{i=1}^{\infty} \alpha_i \boldsymbol{g}_i(m)$ in the asymptotic expansion of (1.1) does not need to be convergent; it may diverge in general. Finally, the α_i are not all nonzero necessarily; some may be zero in general.[1]

Clearly, if $\alpha_1 \neq 0$ and $\lim_{m\to\infty} \boldsymbol{g}_1(m) = \boldsymbol{0}$, then $\{\boldsymbol{x}_m\}$ converges and we have $\lim_{m\to\infty} \boldsymbol{x}_m = \boldsymbol{s}$. If $\alpha_1 \neq 0$ and $\lim_{m\to\infty} \boldsymbol{g}_1(m)$ does not exist, then $\{\boldsymbol{x}_m\}$ diverges. In case it converges, the convergence of the sequence $\{\boldsymbol{x}_m\}$ can be accelerated via a suitable extrapolation method, which will produce good approximations to $\boldsymbol{s}$. Extrapolation methods can be very useful for obtaining good approximations to $\boldsymbol{s}$ also in case of divergence, at least in some cases.

In this work, we would like to analyze the convergence and acceleration properties of one such method, namely, the *vector E-algorithm* of Brezinski [1]. See also Brezinski and Redivo Zaglia [2, Chapter 4, pp. 228–232]. The vector E-algorithm

[1] We may think of an Euler–Maclaurin expansion that may not be full, for example.

produces from the sequence $\{\boldsymbol{x}_m\}$ and the sequences $\{\boldsymbol{g}_i(m)\}_{m=0}^{\infty}$, $i = 1, 2, \ldots,$ a two-dimensional array of approximations $\boldsymbol{s}_{n,k}$, which are defined via

$$\boldsymbol{s}_{n,k} = \boldsymbol{x}_n - \sum_{i=1}^{k} \widetilde{\alpha}_i \boldsymbol{g}_i(n), \tag{1.4}$$

the $\widetilde{\alpha}_i$ being the solution to the $k \times k$ linear system

$$\sum_{i=1}^{k} \langle \boldsymbol{y}, \Delta\boldsymbol{g}_i(m)\rangle \widetilde{\alpha}_i = \langle \boldsymbol{y}, \Delta\boldsymbol{x}_m\rangle, \quad m = n, n+1, \ldots, n+k-1. \tag{1.5}$$

Here, $\boldsymbol{y}$ is a nonzero vector in $\mathbb{X}$, $\Delta\boldsymbol{x}_m = \boldsymbol{x}_{m+1} - \boldsymbol{x}_m$ and $\Delta\boldsymbol{g}_i(m) = \boldsymbol{g}_i(m+1) - \boldsymbol{g}_i(m)$. Taken together, and by Cramer's rule, (1.4) and (1.5) give rise to the following determinant representation for $\boldsymbol{s}_{n,k}$:

$$\boldsymbol{s}_{n,k} = f_{n,k}(\boldsymbol{x}); \quad \boldsymbol{x} \equiv \{\boldsymbol{x}_m\}, \tag{1.6}$$

where, for an arbitrary vector sequence $\boldsymbol{v} \equiv \{\boldsymbol{v}_m\}$ in $\mathbb{X}$,

$$f_{n,k}(\boldsymbol{v}) = \frac{N_{n,k}(\boldsymbol{v})}{D_{n,k}} = \frac{\begin{vmatrix} \boldsymbol{v}_n & \langle \boldsymbol{y}, \Delta\boldsymbol{v}_n\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{v}_{n+k-1}\rangle \\ \boldsymbol{g}_1(n) & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n+k-1)\rangle \\ \vdots & \vdots & & \vdots \\ \boldsymbol{g}_k(n) & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n+k-1)\rangle \end{vmatrix}}{\begin{vmatrix} \langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n+k-1)\rangle \\ \vdots & & \vdots \\ \langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n+k-1)\rangle \end{vmatrix}}. \tag{1.7}$$

Of course, we are assuming that $D_{n,k}$, the denominator determinant of $\boldsymbol{s}_{n,k}$, is nonzero. Note also that $N_{n,k}(\boldsymbol{x})$, the numerator determinant of $\boldsymbol{s}_{n,k}$, which is a vector, is to be interpreted as its expansion with respect to its first column.

A recursion relation for the $\boldsymbol{s}_{n,k}$ is given in Brezinski [1]. Different recursion relations for this method are also given in Ford and Sidi [3].

For convenience, let us arrange the $\boldsymbol{s}_{n,k}$ in a two-dimensional array as in Table 1, where $\boldsymbol{s}_{n,0} = \boldsymbol{x}_n$, $n = 0, 1, \ldots$.

2 A convergence theory

Convergence acceleration properties of the rows $\{\boldsymbol{s}_{n,k}\}_{n=0}^{\infty}$, $k = 1, 2, \ldots,$ of the extrapolation table, that is, convergence acceleration properties of $\boldsymbol{s}_{n,k}$ as $n \to \infty$

Table 1 The extrapolation table

$\boldsymbol{s}_{0,0}$	$\boldsymbol{s}_{1,0}$	$\boldsymbol{s}_{2,0}$	$\cdots$
$\boldsymbol{s}_{0,1}$	$\boldsymbol{s}_{1,1}$	$\boldsymbol{s}_{2,1}$	$\cdots$
$\boldsymbol{s}_{0,2}$	$\boldsymbol{s}_{1,2}$	$\boldsymbol{s}_{2,2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

with k fixed, have been considered under different conditions in the works of Wimp [7, Chapter 10, p. 180, Theorem 1] and Matos [4]. Here, we provide a new study, whose results are summarized in Theorem 1 that is stated and proved below. This theorem provides optimal results in the form of

1. a genuine asymptotic expansion for $\boldsymbol{s}_{n,k}$ as $n \to \infty$, and
2. a definitive and quantitative convergence acceleration result.

The technique we use to prove Theorem 1 is derived in part from Wimp [7] and mostly from Sidi [5], with necessary modifications to accommodate vector sequences. It also involves the notion of *generalized asymptotic expansion*; see Temme [6, Chapter 1], for example. For convenience, we give the precise definition of this notion here.

Definition 1 Let $\{\phi_i(m)\}_{i=1}^{\infty}$ and $\{\psi_i(m)\}_{i=1}^{\infty}$ be two asymptotic scales as $m \to \infty$. Let also $\{W_m\}_{m=0}^{\infty}$ be a given sequence. We say that the formal series $\sum_{i=1}^{\infty} a_i\phi_i(m)$ is the *generalized asymptotic expansion of W_m with respect to $\{\psi_i(m)\}_{i=1}^{\infty}$ as $m \to \infty$,* written in the form

$$W_m \sim \sum_{i=1}^{\infty} a_i\phi_i(m) \quad \text{as } m \to \infty; \quad \{\psi_i\},$$

provided

$$W_m - \sum_{i=1}^{r} a_i\phi_i(m) = o(\psi_r(m)) \quad \text{as } m \to \infty, \quad \forall\, r \geq 1.$$

The notation we use in the sequel is precisely that introduced in the previous section.

Theorem 1 *Let the sequence $\{\boldsymbol{x}_m\}$ be as in (1.1), with the $\boldsymbol{g}_i(m)$ satisfying (1.2), and*

$$\begin{gathered}\lim_{m\to\infty} \frac{\langle \boldsymbol{y}, \boldsymbol{g}_i(m+1)\rangle}{\langle \boldsymbol{y}, \boldsymbol{g}_i(m)\rangle} = b_i \neq 1, \quad i = 1, 2, \ldots, \\ b_i \text{ distinct}; \quad |b_1| > |b_2| > \cdots; \quad \lim_{i\to\infty} b_i = 0,\end{gathered} \tag{2.1}$$

in addition. Assume also that

$$\lim_{m\to\infty} \frac{\boldsymbol{g}_i(m)}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_i(m)\rangle} = \widehat{\boldsymbol{g}}_i \neq \boldsymbol{0}, \quad i = 1, 2, \ldots, \tag{2.2}$$

and define

$$\widehat{\boldsymbol{h}}_{k,i} = \begin{vmatrix} \widehat{\boldsymbol{g}}_i & 1 & b_i & \cdots & b_i^{k-1} \\ \widehat{\boldsymbol{g}}_1 & 1 & b_1 & \cdots & b_1^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \widehat{\boldsymbol{g}}_k & 1 & b_k & \cdots & b_k^{k-1} \end{vmatrix}, \quad i \geq k+1. \tag{2.3}$$

Then the following are true:

1. *There holds*

$$\lim_{m\to\infty} \frac{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m+1)\rangle}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle} = b_i, \quad i = 1, 2, \ldots, \tag{2.4}$$

in addition to (2.1). *Furthermore, the sequence* $\{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle\}_{i=1}^{\infty}$ *is an asymptotic scale as* $m \to \infty$, *that is,*

$$\lim_{m\to\infty} \frac{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_{i+1}(m)\rangle}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle} = 0, \quad i = 1, 2, \ldots . \tag{2.5}$$

2. *With arbitrary* $\boldsymbol{v} \equiv \{\boldsymbol{v}_m\}_{m=0}^{\infty}$, $f_{n,k}(\boldsymbol{v})$ *defined in* (1.7) *exist for all* $n \geq n_0$, n_0 *being some positive integer independent of* $\boldsymbol{v}$.
3. (a) *With* $\boldsymbol{g}_i \equiv \{\boldsymbol{g}_i(m)\}_{m=0}^{\infty}$, *we have* $f_{n,k}(\boldsymbol{g}_i) = \boldsymbol{0}$ *for* $i = 1, \ldots, k$, *while for* $i \geq k+1$,

$$\begin{aligned} \frac{f_{n,k}(\boldsymbol{g}_i)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle} &\sim \frac{\widehat{\boldsymbol{h}}_{k,i}}{V(b_1,\ldots,b_k)} \quad \text{as } n \to \infty, \quad \text{if } \widehat{\boldsymbol{h}}_{k,i} \neq \boldsymbol{0}, \\ \frac{f_{n,k}(\boldsymbol{g}_i)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle} &= o(1) \quad \text{as } n \to \infty, \quad \text{if } \widehat{\boldsymbol{h}}_{k,i} = \boldsymbol{0}, \end{aligned} \tag{2.6}$$

and also

$$\begin{aligned} \|f_{n,k}(\boldsymbol{g}_i)\| &\sim C_{k,i}\|\boldsymbol{g}_i(n)\| \quad \text{as } n \to \infty, \quad \text{if } \widehat{\boldsymbol{h}}_{k,i} \neq \boldsymbol{0}, \\ \|f_{n,k}(\boldsymbol{g}_i)\| &= o(\|\boldsymbol{g}_i(n)\|) \quad \text{as } n \to \infty, \quad \text{if } \widehat{\boldsymbol{h}}_{k,i} = \boldsymbol{0}, \end{aligned} \tag{2.7}$$

where

$$C_{k,i} = \frac{1}{|V(b_1,\ldots,b_k)|} \frac{\|\widehat{\boldsymbol{h}}_{k,i}\|}{\|\widehat{\boldsymbol{g}}_i\|} \tag{2.8}$$

and $V(c_1,\ldots,c_k)$ *is the Vandermonde determinant of* $c_1,\ldots,c_k$, *given as in*

$$V(c_1,\ldots,c_k) = \begin{vmatrix} 1 & c_1 & \cdots & c_1^{k-1} \\ 1 & c_2 & \cdots & c_2^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & c_k & \cdots & c_k^{k-1} \end{vmatrix} = \prod_{1\leq i<j\leq k} (c_j - c_i). \tag{2.9}$$

(b) *In addition, for* $i \geq k+1$, $\{f_{n,k}(\boldsymbol{g}_i)\}_{i=k+1}^{\infty}$ *is an asymptotic scale as* $n \to \infty$, *in the following generalized sense:*

$$\lim_{n\to\infty} \frac{\|f_{n,k}(\boldsymbol{g}_{i+1})\|}{\|\boldsymbol{g}_i(n)\|} = 0, \quad i \geq k+1. \tag{2.10}$$

4. $\boldsymbol{s}_{n,k}$ *has a genuine generalized asymptotic expansion with respect to the asymptotic scale* $\{\boldsymbol{g}_i(n)\}_{i=1}^{\infty}$ *as* $n \to \infty$; *namely,*

$$\boldsymbol{s}_{n,k} \sim \boldsymbol{s} + \sum_{i=k+1}^{\infty} \alpha_i f_{n,k}(\boldsymbol{g}_i) \quad \text{as } n \to \infty; \quad \{g_i\}, \tag{2.11}$$

in the sense that

$$\boldsymbol{s}_{n,k} - \boldsymbol{s} - \sum_{i=k+1}^{r} \alpha_i f_{n,k}(\boldsymbol{g}_i) = o(\boldsymbol{g}_r(n)) \quad \text{as } n \to \infty, \quad \forall\, r \geq k+1. \tag{2.12}$$

We also have

$$\boldsymbol{s}_{n,k} - \boldsymbol{s} - \sum_{i=k+1}^{r} \alpha_i f_{n,k}(\boldsymbol{g}_i) = o(f_{n,k}(\boldsymbol{g}_r)) \quad \text{as } n \to \infty, \quad \text{if } \widehat{\boldsymbol{h}}_{k,r} \neq \boldsymbol{0}. \tag{2.13}$$

5. *Let $\alpha_{k+\mu}$ be the first nonzero α_{k+i} with $i \geq k+1$. Then the following are true:*

   (a) $\boldsymbol{s}_{n,k}$ *satisfies*

   $$\boldsymbol{s}_{n,k} - \boldsymbol{s} = O(\boldsymbol{g}_{k+\mu}(n)) \quad \text{as } n \to \infty, \tag{2.14}$$

   and, therefore, also

   $$\boldsymbol{s}_{n,k+j} - \boldsymbol{s} = O(\boldsymbol{g}_{k+\mu}(n)) \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots, k+\mu-1. \tag{2.15}$$

   (b) *We also have*

   $$\boldsymbol{s}_{n,k} - \boldsymbol{s} \sim \alpha_{k+\mu} f_{n,k}(\boldsymbol{g}_{k+\mu}) \quad \text{as } n \to \infty, \quad \text{if } \widehat{\boldsymbol{h}}_{k,k+\mu} \neq \boldsymbol{0}. \tag{2.16}$$

   As a result, provided $\widehat{\boldsymbol{h}}_{k+j,k+\mu} \neq \boldsymbol{0}$, $j = 0, 1, \ldots, \mu-1$, we also have

   $$\boldsymbol{s}_{n,k+j} - \boldsymbol{s} \sim \alpha_{k+\mu} f_{n,k+j}(\boldsymbol{g}_{k+\mu}) \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots, \mu-1, \tag{2.17}$$

   which also implies

   $$\|\boldsymbol{s}_{n,k+j} - \boldsymbol{s}\| \sim |\alpha_{k+\mu}|\, C_{k+j,k+\mu}\, \|\boldsymbol{g}_{k+\mu}(n)\| \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots, \mu-1. \tag{2.18}$$

   (c) *If $\alpha_k \neq 0$ and $\widehat{\boldsymbol{h}}_{k-1,k} \neq \boldsymbol{0}$, then*

   $$\frac{\|\boldsymbol{s}_{n,k+j} - \boldsymbol{s}\|}{\|\boldsymbol{s}_{n,k-1} - \boldsymbol{s}\|} = O\left(\frac{\|\boldsymbol{g}_{k+\mu}(n)\|}{\|\boldsymbol{g}_k(n)\|}\right) = o(1) \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots, k+\mu-1. \tag{2.19}$$

Proof of part 1 We first note that

$$\frac{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m+1)\rangle}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle} = \frac{\langle \boldsymbol{y}, \boldsymbol{g}_i(m+1)\rangle}{\langle \boldsymbol{y}, \boldsymbol{g}_i(m)\rangle} \, \frac{\dfrac{\langle \boldsymbol{y}, \boldsymbol{g}_i(m+2)\rangle}{\langle \boldsymbol{y}, \boldsymbol{g}_i(m+1)\rangle} - 1}{\dfrac{\langle \boldsymbol{y}, \boldsymbol{g}_i(m+1)\rangle}{\langle \boldsymbol{y}, \boldsymbol{g}_i(m)\rangle} - 1}.$$

Taking now limits as $m \to \infty$, and invoking (2.1), we obtain (2.4).

Next, by (2.2), we have the asymptotic equality

$$\boldsymbol{g}_i(m) \sim \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle\, \widehat{\boldsymbol{g}}_i \quad \text{as } m \to \infty, \tag{2.20}$$

which, upon taking norms, gives the asymptotic equality

$$|\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle| \sim \frac{\|\boldsymbol{g}_i(m)\|}{\|\widehat{\boldsymbol{g}}_i\|} \quad \text{as } m \to \infty. \tag{2.21}$$

Therefore,

$$\frac{|\langle \boldsymbol{y}, \Delta \boldsymbol{g}_{i+1}(m)\rangle|}{|\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle|} \sim \frac{\|\widehat{\boldsymbol{g}}_i\|}{\|\widehat{\boldsymbol{g}}_{i+1}\|} \frac{\|\boldsymbol{g}_{i+1}(m)\|}{\|\boldsymbol{g}_i(m)\|} \quad \text{as } m \to \infty.$$

Invoking now the fact that $\{\boldsymbol{g}_i(m)\}_{i=1}^{\infty}$ itself is an asymptotic scale as $m \to \infty$, as in (1.2), the result in (2.5) follows. □

Proof of part 2 By (1.7), $f_{n,k}(\boldsymbol{v})$ for arbitrary $\boldsymbol{v} \equiv \{\boldsymbol{v}_m\}$ exists provided $D_{n,k}$, the denominator determinant, is nonzero. Therefore, we need to analyze only the determinant $D_{n,k}$ in (1.7). Let us set

$$\eta_{i,j}(m) = \frac{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m+j)\rangle}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m)\rangle}, \quad i, j = 1, 2, \ldots, \tag{2.22}$$

and observe that

$$\eta_{i,j}(m) = \prod_{r=1}^{j} \frac{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m+r)\rangle}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(m+r-1)\rangle}. \tag{2.23}$$

Letting $m \to \infty$ and invoking (2.4), we obtain

$$\lim_{m\to\infty} \eta_{i,j}(m) = b_i^j. \tag{2.24}$$

Factoring out $\langle \boldsymbol{y}, \Delta \boldsymbol{g}_j(n)\rangle$ from the jth row of $D_{n,k}$, $j = 1, \ldots, k$, we have

$$\frac{D_{n,k}}{\prod_{j=1}^{k}\langle \boldsymbol{y}, \Delta \boldsymbol{g}_j(n)\rangle} = \begin{vmatrix} 1 & \eta_{1,1}(n) & \cdots & \eta_{1,k-1}(n) \\ \vdots & \vdots & & \vdots \\ 1 & \eta_{k,1}(n) & \cdots & \eta_{k,k-1}(n) \end{vmatrix} \equiv \psi_{n,k}, \tag{2.25}$$

which, upon letting $n \to \infty$, gives

$$\lim_{n\to\infty} \psi_{n,k} = V(b_1, b_2, \ldots, b_k) = \prod_{1\le i<j\le k} (b_j - b_i), \tag{2.26}$$

this limit being nonzero since the b_i are distinct. Therefore,

$$D_{n,k} \sim V(b_1, b_2, \ldots, b_k) \prod_{j=1}^{k} \langle \boldsymbol{y}, \Delta \boldsymbol{g}_j(n)\rangle \quad \text{as } n \to \infty.$$

From this and from (2.21), we conclude that $D_{n,k} \neq 0$ for all large n. Since $D_{n,k}$ is also independent of $\boldsymbol{v}$, we have that $D_{n,k} \neq 0$ for all $n \ge n_0$, n_0 being independent of $\boldsymbol{v}$ trivially. □

Proof of part 3 We now turn to $f_{n,k}(\boldsymbol{g}_i) = N_{n,k}(\boldsymbol{g}_i)/D_{n,k}$, where

$$N_{n,k}(\boldsymbol{g}_i) = \begin{vmatrix} \boldsymbol{g}_i(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n+k-1)\rangle \\ \boldsymbol{g}_1(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n+k-1)\rangle \\ \vdots & \vdots & & \vdots \\ \boldsymbol{g}_k(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n+k-1)\rangle \end{vmatrix}. \tag{2.27}$$

We first observe that $N_{n,k}(\boldsymbol{g}_i) = \boldsymbol{0}$ for $i = 1, \ldots, k$, since the determinant in (2.27) has two identical rows when $1 \leq i \leq k$. This proves that $f_{n,k}(\boldsymbol{g}_i) = \boldsymbol{0}$ for $i = 1, \ldots, k$. Therefore, we consider the case $i \geq k+1$. Proceeding as in the analysis of $D_{n,k}$, let us factor out $\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle$ and $\langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n)\rangle, \ldots, \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n)\rangle$ from the $k+1$ rows of $N_{n,k}(\boldsymbol{g}_i)$. We obtain

$$\frac{N_{n,k}(\boldsymbol{g}_i)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle \prod_{j=1}^{k} \langle \boldsymbol{y}, \Delta \boldsymbol{g}_j(n)\rangle} = \begin{vmatrix} \frac{\boldsymbol{g}_i(n)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle} & 1 & \eta_{i,1}(n) & \cdots & \eta_{i,k-1}(n) \\ \frac{\boldsymbol{g}_1(n)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n)\rangle} & 1 & \eta_{1,1}(n) & \cdots & \eta_{1,k-1}(n) \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\boldsymbol{g}_k(n)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n)\rangle} & 1 & \eta_{k,1}(n) & \cdots & \eta_{k,k-1}(n) \end{vmatrix} \equiv \boldsymbol{h}_{k,i}(n), \tag{2.28}$$

which, upon letting $n \to \infty$ and invoking (2.24), (2.2), and (2.3), gives

$$\lim_{n\to\infty} \boldsymbol{h}_{k,i}(n) = \widehat{\boldsymbol{h}}_{k,i}. \tag{2.29}$$

Combining now (2.28) with (2.25), we obtain

$$f_{n,k}(\boldsymbol{g}_i) = \frac{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle \boldsymbol{h}_{k,i}(n)}{\psi_{n,k}}, \tag{2.30}$$

which, upon letting $n \to \infty$, gives

$$\lim_{n\to\infty} \frac{f_{n,k}(\boldsymbol{g}_i)}{\langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n)\rangle} = \frac{\widehat{\boldsymbol{h}}_{k,i}}{V(b_1, \ldots, b_k)}, \tag{2.31}$$

from which, (2.6) follows. (2.7) is obtained by taking norms in (2.6) and by making use of (2.21).

Finally,

$$\frac{\| f_{n,k}(\boldsymbol{g}_{i+1})\|}{\|\boldsymbol{g}_i(n)\|} = \frac{|\langle \boldsymbol{y}, \Delta \boldsymbol{g}_{i+1}(n)\rangle| \, \|\boldsymbol{h}_{k,i}(n)\|}{|\psi_{n,k}| \, \|\boldsymbol{g}_i(n)\|},$$

which, upon letting $n \to \infty$ and invoking (2.21), gives

$$\frac{\| f_{n,k}(\boldsymbol{g}_{i+1})\|}{\|\boldsymbol{g}_i(n)\|} \sim \frac{1}{|V(b_1, \ldots, b_k)|} \frac{\|\boldsymbol{h}_{k,i}(n)\|}{\|\widehat{\boldsymbol{g}}_{i+1}\|} \frac{\|\boldsymbol{g}_{i+1}(n)\|}{\|\boldsymbol{g}_i(n)\|} \quad \text{as } n \to \infty.$$

Invoking here (1.2), and noting that $\|\boldsymbol{h}_{k,i}(n)\|$ is bounded in n, we obtain (2.10). □

Proof of part 4 We now turn to $\boldsymbol{s}_{n,k}$. First, we note that

$$\boldsymbol{s}_{n,k} - \boldsymbol{s} = f_{n,k}(\boldsymbol{x}) - \boldsymbol{s} = \frac{N_{n,k}(\boldsymbol{x} - \boldsymbol{s})}{D_{n,k}}; \quad \boldsymbol{x} - \boldsymbol{s} \equiv \{\boldsymbol{x}_m - \boldsymbol{s}\}, \tag{2.32}$$

with

$$N_{n,k}(\boldsymbol{x} - \boldsymbol{s}) = \begin{vmatrix} \boldsymbol{x}_n - \boldsymbol{s} & \langle \boldsymbol{y}, \Delta \boldsymbol{x}_n\rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{x}_{n+k-1}\rangle \\ \boldsymbol{g}_1(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n+k-1)\rangle \\ \vdots & \vdots & & \vdots \\ \boldsymbol{g}_k(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n+k-1)\rangle \end{vmatrix}, \tag{2.33}$$

because the coefficient of $\boldsymbol{x}_n$ in the expansion of $N_{n,k}(\boldsymbol{x})$ is $D_{n,k}$. By (1.1), the elements in the first row of $N_{n,k}(\boldsymbol{x}-\boldsymbol{s})$ have the asymptotic expansions

$$\boldsymbol{x}_n - \boldsymbol{s} \sim \sum_{i=1}^{\infty} \alpha_i \boldsymbol{g}_i(n) \quad \text{as } n \to \infty,$$

$$\langle \boldsymbol{y}, \Delta \boldsymbol{x}_{n+j} \rangle \sim \sum_{i=1}^{\infty} \alpha_i \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n+j) \rangle \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots .$$

Multiplying the $(i+1)$st row of $N_{n,k}(\boldsymbol{x}-\boldsymbol{s})$ in (2.33) by α_i and subtracting from the first row, $i = 1, \ldots, k$, we obtain

$$N_{n,k}(\boldsymbol{x}-\boldsymbol{s}) \sim \begin{vmatrix} \sum_{i=k+1}^{\infty} \alpha_i \boldsymbol{g}_i(n) & \sum_{i=k+1}^{\infty} \alpha_i \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n) \rangle & \cdots & \sum_{i=k+1}^{\infty} \alpha_i \langle \boldsymbol{y}, \Delta \boldsymbol{g}_i(n+k-1) \rangle \\ \boldsymbol{g}_1(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n) \rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_1(n+k-1) \rangle \\ \vdots & \vdots & & \vdots \\ \boldsymbol{g}_k(n) & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n) \rangle & \cdots & \langle \boldsymbol{y}, \Delta \boldsymbol{g}_k(n+k-1) \rangle \end{vmatrix} \tag{2.34}$$

as $n \to \infty$. Taking the summations $\sum_{i=k+1}^{\infty}$ and the multiplicative factors α_i from the first row outside the determinant in (2.34), we have

$$N_{n,k}(\boldsymbol{x}-\boldsymbol{s}) \sim \sum_{i=k+1}^{\infty} \alpha_i N_{n,k}(\boldsymbol{g}_i) \quad \text{as } n \to \infty; \quad \boldsymbol{g}_i \equiv \{\boldsymbol{g}_i(m)\}_{m=0}^{\infty}, \tag{2.35}$$

with $N_{n,k}(\boldsymbol{g}_i)$ as in (2.27). Substituting (2.35) in (2.32), we obtain the asymptotic expansion of $\boldsymbol{s}_{n,k}$ given in (2.11). This asymptotic expansion will be a valid generalized asymptotic expansion with respect to the asymptotic scale $\{\boldsymbol{g}_i(n)\}_{i=1}^{\infty}$ as $n \to \infty$, provided

$$\boldsymbol{s}_{n,k} - \boldsymbol{s} - \sum_{i=k+1}^{r} \alpha_i f_{n,k}(\boldsymbol{g}_i) = o(\boldsymbol{g}_r(n)) \quad \text{as } n \to \infty, \quad \forall\, r \geq k+1. \tag{2.36}$$

By (1.3), for arbitrary r, we have

$$\boldsymbol{x}_m = \boldsymbol{s} + \sum_{i=1}^{r} \alpha_i \boldsymbol{g}_i(m) + \boldsymbol{\epsilon}_r(m); \quad \boldsymbol{\epsilon}_r(m) = o(\boldsymbol{g}_r(m)) \quad \text{as } m \to \infty. \tag{2.37}$$

Let us substitute this in (2.32) and proceed exactly as above; we obtain

$$\boldsymbol{s}_{n,k} = \boldsymbol{s} + \sum_{i=k+1}^{r} \alpha_i f_{n,k}(\boldsymbol{g}_i) + f_{n,k}(\boldsymbol{\epsilon}_r). \tag{2.38}$$

Comparing (2.38) with (2.36), we realize that (2.36) will be satisfied provided

$$f_{n,k}(\boldsymbol{\epsilon}_r) = o(\boldsymbol{g}_r(n)) \quad \text{as } n \to \infty. \tag{2.39}$$

Now, $f_{n,k}(\boldsymbol{\epsilon}_r) = N_{n,k}(\boldsymbol{\epsilon}_r)/D_{n,k}$, and

$$N_{n,k}(\boldsymbol{\epsilon}_r) = \begin{vmatrix} \boldsymbol{\epsilon}_r(n) & \langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n+k-1)\rangle \\ \boldsymbol{g}_1(n) & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n+k-1)\rangle \\ \vdots & \vdots & & \vdots \\ \boldsymbol{g}_k(n) & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n)\rangle & \cdots & \langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n+k-1)\rangle \end{vmatrix}.$$

Let us factor out $\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle$ and $\langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n)\rangle,\ldots,\langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n)\rangle$ from the $k+1$ rows of this determinant. We obtain

$$\frac{N_{n,k}(\boldsymbol{\epsilon}_r)}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle \prod_{j=1}^{k}\langle \boldsymbol{y}, \Delta\boldsymbol{g}_j(n)\rangle} =$$

$$\begin{vmatrix} \frac{\boldsymbol{\epsilon}_r(n)}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle} & \frac{\langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n)\rangle}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle} & \frac{\langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n+1)\rangle}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle} & \cdots & \frac{\langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n+k-1)\rangle}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle} \\ \frac{\boldsymbol{g}_1(n)}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_1(n)\rangle} & 1 & \eta_{1,1}(n) & \cdots & \eta_{1,k-1}(n) \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\boldsymbol{g}_k(n)}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_k(n)\rangle} & 1 & \eta_{k,1}(n) & \cdots & \eta_{k,k-1}(n) \end{vmatrix} \equiv \phi_{n,k}(\boldsymbol{\epsilon}_r). \quad (2.40)$$

Dividing now (2.40) by (2.25), we obtain

$$f_{n,k}(\boldsymbol{\epsilon}_r) = \frac{\phi_{n,k}(\boldsymbol{\epsilon}_r)}{\psi_{n,k}}\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle,$$

which, upon taking norms and invoking (2.21), gives

$$\|f_{n,k}(\boldsymbol{\epsilon}_r)\| \sim \frac{\|\phi_{n,k}(\boldsymbol{\epsilon}_r)\|}{|V(b_1,\ldots,b_k)|}\frac{\|\boldsymbol{g}_r(n)\|}{\|\widehat{\boldsymbol{g}}_r\|} \quad \text{as } n\to\infty.$$

Therefore, (2.36) will hold provided $\lim_{n\to\infty}\phi_{n,k}(\boldsymbol{\epsilon}_r) = 0$. As we already know, with the exception of the elements in the first row, all the remaining elements of the determinant $\phi_{n,k}(\boldsymbol{\epsilon}_r)$ have finite limits as $n\to\infty$, by (2.2) and (2.24). Therefore, $\lim_{n\to\infty}\phi_{n,k}(\boldsymbol{\epsilon}_r) = \mathbf{0}$ will hold provided all the elements in the first row of $\phi_{n,k}(\boldsymbol{\epsilon}_r)$ tend to zero as $n\to\infty$. That this is the case is what we show next.

First, by (2.21)–(2.24), as $n\to\infty$,

$$\|\boldsymbol{g}_r(n+j)\| \sim \|\widehat{\boldsymbol{g}}_r\|\,|\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n+j)\rangle| \sim |b_r^j|\,\|\widehat{\boldsymbol{g}}_r\|\,|\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle| \sim |b_r^j|\,\|\boldsymbol{g}_r(n)\|. \quad (2.41)$$

Next, by applying the Cauchy–Schwarz inequality to $\langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n+j)\rangle$, and invoking (2.37) and (2.41), we have

$$\begin{aligned} |\langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n+j)\rangle| &\le \|\boldsymbol{y}\|\,(\|\boldsymbol{\epsilon}_r(n+j+1)\| + \|\boldsymbol{\epsilon}_r(n+j)\|) \\ &= o(\|\boldsymbol{g}_r(n+j+1)\|) + o(\|\boldsymbol{g}_r(n+j)\|) \quad \text{as } n\to\infty \\ &= o(\|\boldsymbol{g}_r(n)\|) \quad \text{as } n\to\infty. \end{aligned}$$

Invoking also (2.21), for the elements in the first row of $\phi_{n,k}(\boldsymbol{\epsilon}_r)$, we finally obtain

$$\left\|\frac{\boldsymbol{\epsilon}_r(n)}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle}\right\| = o(1) \quad \text{as } n\to\infty,$$

and

$$\left|\frac{\langle \boldsymbol{y}, \Delta\boldsymbol{\epsilon}_r(n+j)\rangle}{\langle \boldsymbol{y}, \Delta\boldsymbol{g}_r(n)\rangle}\right| = o(1) \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots, k-1.$$

This implies that $\phi_{n,k}(\boldsymbol{\epsilon}_r) = o(1)$ as $n \to \infty$, and the proof is complete. □

Proof of Part 5 By $\alpha_{k+j} = 0$, $j = 1, \ldots, \mu - 1$, and $\alpha_{k+\mu} \neq 0$,[2] the validity of (2.14) is obvious. (2.15) follows from (2.14). The validity of (2.16)–(2.18) can be shown in the same way. As for (2.19), we start with

$$\frac{\|\boldsymbol{s}_{n,k} - \boldsymbol{s}\|}{\|\boldsymbol{s}_{n,k-1} - \boldsymbol{s}\|} \sim \frac{\|\boldsymbol{s}_{n,k} - \boldsymbol{s}\|}{|\alpha_k| C_{k-1,k} \|\boldsymbol{g}_k(n)\|},$$

which follows from (2.18) and invoke (2.15). We leave the details to the reader. □

3 Remarks on the convergence theory

1. Note that Theorem 1 is stated under a minimal number of conditions on the $\boldsymbol{g}_i(m)$ and the $\boldsymbol{x}_m$. Of these, the condition in (2.1) is already in [7, p. 180, Eq.(3)], while that in (2.2) is a modification of [7, p. 180, Eq.(5)].
2. The conditions we have imposed on the $\boldsymbol{g}_i(m)$ enable us to proceed with the proof rigorously by employing asymptotic equalities $\sim$ everywhere possible. This should be contrasted with bounds formulated in terms of the big O notation, which do not allow us to obtain the optimal results we have in our theorem.[3]
3. Note that we have imposed essentially two different conditions on the $\boldsymbol{g}_i(m)$, namely (2.1) and (2.2). One may naturally think that these conditions could contradict each other. In addition, one may think that they could also contradict the very first and fundamental property in (1.2), which must hold to make (1.1) a genuine asymptotic expansion. Thus, we need to make sure that there are no contradictions present in our theorem. For this, it is enough to show that all three conditions can hold simultaneously, which is the case when

$$\boldsymbol{g}_i(m) \sim \boldsymbol{w}_i b_i^m \quad \text{as } m \to \infty, \quad |b_i| > |b_{i+1}| \quad \forall\, i \geq 1.$$

   It is easy to verify that (1.2), (2.1), and (2.2) are satisfied simultaneously in this case.
4. Due to the possibility that $\widehat{\boldsymbol{h}}_{k,i} = \boldsymbol{0}$ for some $i \geq k+1$, we cannot claim a priori that $\{f_{n,k}(\boldsymbol{g}_i)\}_{i=k+1}^{\infty}$ is an asymptotic scale in the regular sense. Note, however, that we can safely replace (2.6) by

$$\|f_{n,k}(\boldsymbol{g}_i)\| = O(\|\boldsymbol{g}_i(n)\|) \quad \text{as } n \to \infty, \quad \forall\, i \geq k+1,$$

   whether $\widehat{\boldsymbol{h}}_{k,i} \neq \boldsymbol{0}$ or $\widehat{\boldsymbol{h}}_{k,i} = \boldsymbol{0}$.

[2] Note that this already takes into account the possibility that $\alpha_{k+1} \neq 0$, in which case, $\mu = 1$.

[3] Recall that $u_m \sim v_m$ as $m \to \infty$ if and only if $\lim_{m\to\infty}(u_m/v_m) = 1$. One big advantage of asymptotic equalities is that they allow symmetry and division. That is, if $u_m \sim v_m$ then $v_m \sim u_m$ as well. In addition, $u_m \sim v_m$ and $u'_m \sim v'_m$ also imply $u_m/u'_m \sim v_m/v'_m$. On the other hand, if $u_m = O(v_m)$, we do not necessarily have $v_m = O(u_m)$. In addition, $u_m = O(v_m)$ and $u'_m = O(v'_m)$ do not necessarily imply $u_m/u'_m = O(v_m/v'_m)$.

5. $\widehat{\boldsymbol{h}}_{k,i} \neq \mathbf{0}$ for all $i \geq k+1$ if, for example, the vectors $\widehat{\boldsymbol{g}}_i$ are all linearly independent, which is possible if $\mathbb{X}$ is an infinite dimensional space. This can be seen by expanding the determinant defining $\widehat{\boldsymbol{h}}_{k,i}$ in (2.3) with respect to its first column and realizing that $\widehat{\boldsymbol{h}}_{k,i} = c_i\widehat{\boldsymbol{g}}_i + \sum_{j=1}^{k} c_j\widehat{\boldsymbol{g}}_j$, where c_i and the c_j are all nonzero Vandermonde determinants. In such a case, by (2.7) and (1.2),

$$\frac{\|f_{n,k}(\boldsymbol{g}_{i+1})\|}{\|f_{n,k}(\boldsymbol{g}_i)\|} \sim \frac{C_{k,i+1}}{C_{k,i}} \frac{\|\boldsymbol{g}_{i+1}(n)\|}{\|\boldsymbol{g}_i(n)\|} = o(1) \quad \text{as } n \to \infty,$$

hence $\{f_{n,k}(\boldsymbol{g}_i)\}_{i=1}^{\infty}$ is an asymptotic scale in the regular sense. Therefore, the asymptotic expansion of $\boldsymbol{s}_{n,k}$ in (2.11) is a *regular* asymptotic expansion, which means that

$$\boldsymbol{s}_{n,k} - \boldsymbol{s} - \sum_{i=k+1}^{r} \alpha_i f_{n,k}(\boldsymbol{g}_i) = o(f_{n,k}(\boldsymbol{g}_r)) \quad \text{as } n \to \infty, \quad \forall\, r \geq k+1.$$

6. When $\alpha_1 \neq 0$, the sequence $\{\boldsymbol{x}_m\}$ is convergent if $|b_1| < 1$; it is divergent if $|b_1| \geq 1$. The asymptotic result in (2.12), which is always true, shows clearly that $\boldsymbol{s}_{n,k}$ converges to $\boldsymbol{s}$ faster than $\boldsymbol{x}_n$ when $\{\boldsymbol{x}_m\}$ is convergent. In case $\{\boldsymbol{x}_m\}$ is divergent, by the assumption that $\lim_{i\to\infty} b_i = 0$, we have that $|b_i| < 1$, $i \geq p$, for some integer p, and $\boldsymbol{s}_{n,k}$ converges when $k \geq p$.
7. Consider the case

$$\alpha_k \neq 0, \quad \alpha_{k+1} = \cdots = \alpha_{k+\mu-1} = 0, \quad \alpha_{k+\mu} \neq 0.$$

By (2.12)–(2.19), the following transpire:

- Whether $\widehat{\boldsymbol{h}}_{k-1,k} = \mathbf{0}$ or $\widehat{\boldsymbol{h}}_{k-1,k} \neq \mathbf{0}$,

$$\boldsymbol{s}_{n,k-1} - \boldsymbol{s} = O(\boldsymbol{g}_k(n)) \quad \text{as } n \to \infty,$$

- Whether $\widehat{\boldsymbol{h}}_{k+j,k+\mu} = \mathbf{0}$ or $\widehat{\boldsymbol{h}}_{k+j,k+\mu} \neq \mathbf{0}$, $0 \leq j \leq \mu-1$,

$$\boldsymbol{s}_{n,k+j} - \boldsymbol{s} = O(\boldsymbol{g}_{k+\mu}(n)) \quad \text{as } n \to \infty, \quad 0 \leq j \leq \mu-1.$$

- If $\widehat{\boldsymbol{h}}_{k-1,k} \neq \mathbf{0}$, then $\boldsymbol{s}_{n,k}$ converges faster (or diverges slower) than $\boldsymbol{s}_{n,k-1}$, that is,

$$\lim_{n\to\infty} \frac{\|\boldsymbol{s}_{n,k} - \boldsymbol{s}\|}{\|\boldsymbol{s}_{n,k-1} - \boldsymbol{s}\|} = 0.$$

- If $\widehat{\boldsymbol{h}}_{k+j,k+\mu} \neq \mathbf{0}$, $\quad 0 \leq j \leq \mu-1$, then

$$\|\boldsymbol{s}_{n,k+j} - \boldsymbol{s}\| \sim M_{k+j}\|\boldsymbol{g}_{k+\mu}(n)\| \quad \text{as } n \to \infty, \quad j = 0, 1, \ldots, \mu-1,$$

for some positive constants M_{k+j}. That is, $\boldsymbol{s}_{n,k}, \boldsymbol{s}_{n,k+1}, \ldots, \boldsymbol{s}_{n,k+\mu-1}$ converge (or diverge) at precisely the same rate.

8. We have assumed that $\mathbb{X}$ is an inner product space only for the sake of simplicity. We can assume $\mathbb{X}$ to be a normed Banach space in general. In this case, we replace $\langle \boldsymbol{y}, \boldsymbol{u}\rangle$ by $Q(\boldsymbol{u})$, where Q is a bounded linear functional on $\mathbb{X}$. With this, the analysis of this section goes through in a straightforward manner.

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