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A de Montessus type convergence study for a vector-valued rational interpolation procedure of epsilon class

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Abstract

In a series of recent publications of the author, three rational interpolation methods, denoted IMPE, IMMPE, and ITEA, were proposed for vector-valued functions $F(z)$, where $F: \mathbb{C} \to \mathbb{C}^N$, and their algebraic properties were studied. The convergence studies of two of the methods, namely, IMPE and IMMPE, were also carried out as these methods are being applied to meromorphic functions with simple poles, and de Montessus and König type theorems for them were proved. In the present work, we concentrate on ITEA. We study its convergence properties as it is applied to meromorphic functions with simple poles and prove de Montessus and König type theorems analogous to those obtained for IMPE and IMMPE.

Keywords: vector-valued rational interpolation, Hermite interpolation, Newton interpolation formula, de Montessus theorem, König theorem.

MSC: 30E10, 30E15, 41A20, 41A25.

§1. Introduction and background

In [5], the author developed three rational interpolation methods for vector-valued functions of a complex variable. These methods were denoted IMPE, IMMPE and

ITEA[1]. Each of these methods generates a two-dimensional sequence $\{R_{p,k}(z)\}$ of approximations to a vector-valued function $F(z)$, such that $R_{p,k}(z) = U_{p,k}(z)/V_{p,k}(z)$, where $U_{p,k}(z)$ is a vector-valued polynomial of degree at most $p-1$ and $V_{p,k}(z)$ is a scalar-valued polynomial of degree k, and $R_{p,k}(z)$ interpolates $F(z)$ in the Hermite sense at p points in the complex plane.

Some of the algebraic properties of all these three methods were already presented in [5] while others were explored in [6], where it was also shown that (i) the $R_{p,k}(z)$ are symmetric functions of the points of interpolation and (ii) they reproduce vector-valued rational functions exactly.

In order to be considered valid approximations, in addition to possessing these two algebraic properties, the $R_{p,k}(z)$ should at least allow sound convergence theories of de Montessus and König types when $F(z)$ is a vector-valued meromorphic function, analogously to Padé approximants. Roughly speaking, a de Montessus type theory concerns the convergence properties, in a set Ω of the complex plane, of $R_{p,k}(z)$ as $p \to \infty$ while k is being held fixed, k being the number of the poles (counting multiplicities) of $F(z)$ in Ω. The König type theory concerns the convergence as $p \to \infty$ of the poles of $R_{p,k}(z)$ to those of $F(z)$ in Ω.

In [7, 8, 9], we presented de Montessus and König type convergence theories for IMMPE and IMPE, as these methods are applied to vector-valued meromorphic functions with simple poles. In this work, we treat the convergence properties of ITEA, as it is being applied to the same class of functions, and we prove de Montessus and König type theorems analogous to those for IMPE and IMMPE. As it will become clear, following some necessary adjustments, the techniques of [7] that were developed for analyzing IMMPE will be directly applicable when analyzing ITEA.

Before we go on, we would like to note that the de Montessus type theories developed in the works [7, 8, 9] and in the present work as well, are in the spirit of that developed originally by Saff [3].

§2. Review of the algebraic properties of ITEA

To set the stage for later developments, and to fix the notation as well, we start with a brief description of the developments in [5] and [6].

Let z be a complex variable and let $F(z)$ be a vector-valued function such that $F : \mathbb{C} \to \mathbb{C}^N$. Assume that $F(z)$ is defined on a bounded open set $\Omega \subset \mathbb{C}$ and consider the problem of interpolating $F(z)$ at some of the points $\xi_1, \xi_2, \ldots$ in this set. We do not assume that the ξ_i are necessarily distinct. The general picture is described in the next paragraph.

[1] MPE, MMPE and TEA are three vector extrapolation methods used in accelerating the convergence of sequences of vectors, which have been studied and applied extensively. IMPE, IMMPE and ITEA are interpolatory analogues of MPE, RRE and TEA. The letter "I" in these names stands for the word "interpolatory". For a detailed treatment of vector extrapolation methods and their applications, see the recent book by Sidi [10], for example.

Let $a_1, a_2, \ldots$ be distinct complex numbers and order the ξ_i such that

$$\begin{aligned}
&\xi_1 = \xi_2 = \cdots = \xi_{r_1} = a_1, \\
&\xi_{r_1+1} = \xi_{r_1+2} = \cdots = \xi_{r_1+r_2} = a_2, \\
&\xi_{r_1+r_2+1} = \xi_{r_1+r_2+2} = \cdots = \xi_{r_1+r_2+r_3} = a_3, \\
&\cdots
\end{aligned} \tag{2.1}$$

Let $G_{m,n}(z)$ be the vector-valued polynomial (of degree at most $n-m$) that interpolates $F(z)$ at the points $\xi_m, \xi_{m+1}, \ldots, \xi_n$ in the generalized Hermite sense. Thus, in Newtonian form, this polynomial is given as in (see, e.g., Stoer and Bulirsch [11, Chapter 2] or Atkinson [1, Chapter 3])

$$G_{m,n}(z) = \sum_{i=m}^{n} F[\xi_m, \xi_{m+1}, \ldots, \xi_i] \prod_{j=m}^{i-1} (z - \xi_j). \tag{2.2}$$

Here, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ is the divided difference of order s of $F(z)$ over the set of points $\{\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}\}$. Obviously, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ are all vectors in $\mathbb{C}^N$.

Let us define the scalar polynomials $\psi_{m,n}(z)$ via

$$\psi_{m,n}(z) = \prod_{r=m}^{n} (z - \xi_r), \quad n \geq m \geq 1; \quad \psi_{m,m-1}(z) \equiv 1, \quad m \geq 1.$$

Let us also define the vectors $D_{m,n}$ via

$$D_{m,n} = F[\xi_m, \xi_{m+1}, \ldots, \xi_n], \quad n \geq m.$$

With this notation, we can rewrite (2.2) in the form

$$G_{m,n}(z) = \sum_{i=m}^{n} D_{m,i}\, \psi_{m,i-1}(z).$$

Then the vector-valued rational function $R_{p,k}(z)$ from ITEA that interpolates $F(z)$ at $\xi_1, \ldots, \xi_p$ in the sense of Hermite is defined as

$$R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)} = \frac{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)}, \tag{2.3}$$

the scalars $c_0, c_1, \ldots, c_k$ being determined by the requirement

$$\left(q, \sum_{j=0}^{k} c_j D_{j+1,p+i}\right) = 0, \quad i = 1, \ldots, k; \quad c_k = 1, \tag{2.4}$$

where $(\cdot\,,\cdot)$ is an inner product and q is some fixed nonzero vector in $\mathbb{C}^N$. Clearly, (2.4) results in the linear system

$$\sum_{j=0}^{k-1} u_{i,j} c_j = -u_{i,k}, \quad i = 1, \ldots, k; \quad c_k = 1; \quad u_{i,j} = \big(q, D_{j+1,p+i}\big), \tag{2.5}$$

a unique solution for which exists provided

$$\begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} \neq 0.$$

Combining (2.3) and (2.5), we obtain the following determinant representation for $R_{p,k}(z)$ from ITEA, with $u_{i,j} = (q, D_{j+1,p+i})$, $i \geq 1$, $j \geq 0$:

$$R_{p,k}(z) = \frac{P(z)}{Q(z)} = \frac{\begin{vmatrix} \psi_{1,0}(z)\, G_{1,p}(z) & \psi_{1,1}(z)\, G_{2,p}(z) & \cdots & \psi_{1,k}(z)\, G_{k+1,p}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}{\begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}. \tag{2.6}$$

Here, the numerator determinant $P(z)$ is vector-valued and is defined by its expansion with respect to its first row. That is, if M_j is the cofactor of the term $\psi_{1,j}(z)$ in the denominator determinant $Q(z)$, then

$$R_{p,k}(z) = \frac{\sum_{j=0}^{k} M_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} M_j\, \psi_{1,j}(z)}. \tag{2.7}$$

Note that this determinant representation offers a very effective tool for the algebraic and analytical study of $R_{p,k}(z)$. As we will see later in this work, it forms the basis of our convergence study.

From (2.3) and (2.4), it is clear that the number of function evaluations (namely, (i) $F(\xi_i)$ in case the ξ_i are distinct and (ii) $F(\xi_i)$ and some of its derivatives otherwise) that are needed to determine $R_{p,k}(z)$ is $p+k$, and these are based on $\xi_1,\ldots,\xi_{p+k}$. This should be contrasted with the interpolants $R_{p,k}(z)$ that result from IMPE and IMMPE, which need $p+1$ function evaluations based on $\xi_1,\ldots,\xi_{p+1}$.

Remarks 2.1.

1. $R_{p,k}(z) = U_{p,k}(z)/V_{p,k}(z)$ from ITEA interpolates $F(z)$ at $\xi_1,\ldots,\xi_p$ in the sense of Hermite, provided $V_{p,k}(\xi_i) \neq 0$ for all $i = 1,\ldots,p$.

2. Note that $R_{p,k}(z)$, even with *arbitrary* c_j in (2.3), interpolates $F(z)$ at $\xi_1,\ldots,\xi_p$ in the sense of Hermite, provided $V_{p,k}(\xi_i) \neq 0$ for all $i = 1,\ldots,p$. However, the quality of $R_{p,k}(z)$ as an approximation to $F(z)$ in the z-plane depends heavily on how the c_j are chosen. Thus, the methods IMPE, IMMPE and ITEA choose the c_j in special ways; as we have shown in [7, 8, 9], the methods IMPE and IMMPE do provide very good approximations for meromorphic functions $F(z)$. Here we prove that ITEA does too.

We end this section by stating four algebraic properties of ITEA. Of these, the first three were explored in [6], while the forth is new:

1. *Limiting property:* when all ξ_i tend to 0 simultaneously, it follows from the equations in (2.5) that $R_{p,k}(z)$ tends to the approximant $s_{n+k,k}(z)$ from the method denoted STEA in [4], as the latter is being applied to the Maclaurin series of $F(z)$ [2]. Here $n = p - k$.

2. *Symmetry property:* the denominator polynomial $V_{p,k}(z) = \sum_{j=0}^{k} c_j\psi_{1,j}(z)$ is a symmetric function of $\xi_1,\ldots,\xi_{p+k}$, which go into its construction. $R_{p,k}(z)$ itself is a symmetric function of $\xi_1,\ldots,\xi_p$ [3].

3. *Reproducing property:* if $F(z) = \widetilde{U}(z)/\widetilde{V}(z)$ is a vector-valued rational function with degree of numerator $\widetilde{U}(z)$ at most $p-1$ and degree of denominator $\widetilde{V}(z)$ equal to k and if $F(\xi_i)$, $i = 1,\ldots,p$, are all defined, then $R_{p,k}(z) \equiv F(z)$.

[2] We note that the STEA approximants were developed originally by Brezinski in [2]; they were obtained by applying the topological epsilon algorithm (TEA), which was also developed in [2], to the sequence of partial sums of the Maclaurin series of $F(z)$. See also [10].

[3] A function $f(x_1,\ldots,x_m)$ is symmetric in $x_1,\ldots,x_m$ if $f(x_{i_1},\ldots,x_{i_m}) = f(x_1,\ldots,x_m)$ for every permutation $(x_{i_1},\ldots,x_{i_m})$ of $(x_1,\ldots,x_m)$.

4. *Projection property:* in addition to interpolating $F(z)$ at $\xi_1,\ldots,\xi_p$, $R_{p,k}(z)$ also has the following projection property:
$$\big(q, F(z) - R_{p,k}(z)\big)\big|_{z=\xi_{p+i}} = 0, \quad i = 1,\ldots,k.$$

Next lemma that concerns the scalar case of $N = 1$ provides further justification of our formulation of ITEA as a valid vector-valued rational interpolation procedure.

Lemma 2.2 ([5, Lemma 3.2])**.** *For $N = 1$, that is, when $F(z)$ is a scalar function, $R_{p,k}(z)$ from ITEA interpolates $F(z)$ at the points $\xi_1, \xi_2, \ldots, \xi_{p+k}$ when we take $(q, D_{m,s}) \equiv D_{m,s}$. Thus, $R_{p,k}(z)$ is simply the solution to the Cauchy-Jacobi interpolation problem in this case.*

Because ITEA and IMMPE, in producing the relevant $R_{p,k}(z)$, differ substantially (i) in the number of the ξ_i they use and (ii) in the structure of the relevant scalars $u_{i,j}$, it seems that their analyses should be different from each other. Fortunately, in this work, we are able to overcome these obstacles and apply to ITEA the techniques used for analyzing IMMPE, following some clever adjustments.

To keep things simple, in the sequel, we adopt the notation of [7], where we treated IMMPE. In order not to repeat the arguments of [7] unnecessarily, we will keep our treatment of ITEA short and will refer the reader to [7] for technical details.

§3. Technical preliminaries and error formula when $F(z)$ is a vector-valued rational function

We start our study of ITEA for the case in which the function $F(z)$ is a vector-valued rational function with simple poles, namely,
$$F(z) = \sum_{s=1}^{\mu} \frac{v_s}{z - z_s} + u(z), \tag{3.1}$$
where $u(z)$ is an arbitrary vector-valued polynomial, $z_1,\ldots,z_\mu$ are distinct points in the complex plane, and $v_1,\ldots,v_\mu$ are some nonzero vectors in $\mathbb{C}^N$. An example of such functions is $F(z) = (I - zA)^{-1}b$, where I is the $N \times N$ identity matrix, A is an arbitrary $N \times N$ matrix and b is an N-vector. See [7, Section 3, Example].

3.1. Technical preliminaries

The following technical tools that were used in [7] will be used throughout this work too.

Lemma 3.1 ([7, Lemma 3.2]). *Let* $Q_i(x) = \sum_{j=0}^{i} a_{ij}x^j$, *with* $a_{ii} \neq 0$, $i = 0, 1 \dots, n$, *and let* x_i, $i = 0, 1, \dots, n$, *be arbitrary complex numbers. Then*

$$\begin{vmatrix} Q_0(x_0) & Q_0(x_1) & \cdots & Q_0(x_n) \\ Q_1(x_0) & Q_1(x_1) & \cdots & Q_1(x_n) \\ \vdots & \vdots & & \vdots \\ Q_n(x_0) & Q_n(x_1) & \cdots & Q_n(x_n) \end{vmatrix} = \left(\prod_{i=0}^{n} a_{ii}\right) V(x_0, x_1, \dots, x_n),$$

where

$$V(x_0, x_1, \dots, x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix} = \prod_{0 \le i < j \le n} (x_j - x_i)$$

is a Vandermonde determinant.

Lemma 3.2 ([7, Lemma 3.3]). *Let* $\omega_a(z) = (z - a)^{-1}$. *Then* $\omega_a[\xi_m, \dots, \xi_n]$, *the divided difference of* $\omega_a(z)$ *over the set of points* $\{\xi_m, \dots, \xi_n\}$, *is given by*

$$\omega_a[\xi_m, \dots, \xi_n] = -\frac{1}{\psi_{m,n}(a)}.$$

This is true whether the ξ_i *are distinct or not.*

Lemma 3.3 ([7, Lemma 3.4]). *Let* $F(z)$ *be given as in* (3.1). *Let* $n - m > \deg(u)$. *Then the following are true whether the* ξ_i *are distinct or not:*

(i) $D_{m,n} = F[\xi_m, \dots, \xi_n]$ *is given as*

$$D_{m,n} = -\sum_{s=1}^{\mu} \frac{v_s}{\psi_{m,n}(z_s)}.$$

Therefore, we also have

$$(q, D_{m,n}) = -\sum_{s=1}^{\mu} \frac{(q, v_s)}{\psi_{m,n}(z_s)}.$$

(ii) $F(z) - G_{m,n}(z) = \psi_{m,n}(z)F[z, \xi_m, \dots, \xi_n]$ *is given as*

$$F(z) - G_{m,n}(z) = \psi_{m,n}(z) \sum_{s=1}^{\mu} \frac{v_s}{z - z_s} \frac{1}{\psi_{m,n}(z_s)}. \tag{3.2}$$

3.2. Error formula

Using (2.6), (2.7) and (3.2), we can derive a determinant representation for the error $F(z) - R_{p,k}(z)$ as in the next lemma.

Lemma 3.4 ([7, Lemma 3.5])**.** *Let*

$$\Delta_j(z) = \psi_{1,j}(z)\big[F(z) - G_{j+1,p}(z)\big] = \psi_{1,p}(z)F[z,\xi_{j+1},\ldots,\xi_p], \quad j = 0,1,\ldots \tag{3.3}$$

Then the error in $R_{p,k}(z)$ has the determinant representation

$$F(z) - R_{p,k}(z) = \frac{\Delta(z)}{Q(z)}, \tag{3.4}$$

where

$$\Delta(z) = \begin{vmatrix} \Delta_0(z) & \Delta_1(z) & \cdots & \Delta_k(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}, \quad Q(z) = \begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}. \tag{3.5}$$

We next specialize Lemma 3.3 to suit the error formula for ITEA.

Lemma 3.5. *Let $p > k + \deg u$. Define*

$$\Psi_p(z) \equiv \psi_{1,p+k}(z). \tag{3.6}$$

Then the following are true whether the ξ_i are distinct or not:

(i) $D_{j+1,p+i}$ *is given as*

$$D_{j+1,p+i} = -\sum_{s=1}^{\mu} v_s \psi_{p+i+1,p+k}(z_s)\frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}.$$

Therefore, we also have

$$u_{i,j} = (q, D_{j+1,p+i}) = -\sum_{s=1}^{\mu} \alpha_{i,s}\frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}; \quad \alpha_{i,s} = (q,v_s)\psi_{p+i+1,p+k}(z_s). \tag{3.7}$$

(ii) *As for* $\Delta_j(z)$ *in* (3.3), *we have*

$$\Delta_j(z) = \psi_{1,p}(z) \sum_{s=1}^{\mu} \widehat{e}_s^{(p)}(z) \frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}; \quad \widehat{e}_s^{(p)}(z) = \frac{v_s}{z - z_s} \psi_{p+1,p+k}(z_s). \tag{3.8}$$

Comparing $\Psi_p(z)$ in (3.6), $u_{i,j}$ in (3.7) and $\Delta_j(z)$ in (3.8) with the analogous quantities for IMMPE in [7], we realize that they have the *same* algebraic structure[4]. Therefore, we can now apply the techniques of [7] verbatim, subject to suitable conditions having to do with ITEA.

3.3. Algebraic structures of $Q(z)$, $\Delta(z)$ and $F(z) - R_{p,k}(z)$

Below, we recall that $\Psi_p(z)$ is as in (3.6), $u_{i,j}$ and $\alpha_{i,s}$ are as in (3.7), and $\Delta_j(z)$ and $\widehat{e}_s^{(p)}(z)$ are as in (3.8). Applying theorems 3.6, 3.7 and 3.8 of [7] verbatim to $Q(z)$, $\Delta(z)$ and $F(z) - R_{p,k}(z)$, respectively, we have the following result.

Theorem 3.6 ([7, Theorem 3.6]). *Let* $F(z)$ *be the vector-valued rational function in* (3.1), *precisely as described in the first paragraph of this section, with the notation therein. Define*

$$T_{s_1,\ldots,s_k} = \begin{vmatrix} \alpha_{1,s_1} & \alpha_{1,s_2} & \cdots & \alpha_{1,s_k} \\ \alpha_{2,s_1} & \alpha_{2,s_2} & \cdots & \alpha_{2,s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,s_1} & \alpha_{k,s_2} & \cdots & \alpha_{k,s_k} \end{vmatrix}. \tag{3.9}$$

Then, with $p > k + \deg(u)$,

$$Q(z) = (-1)^k \sum_{1 \le s_1 < s_2 < \cdots < s_k \le \mu} T_{s_1,\ldots,s_k} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left[\prod_{i=1}^{k} \Psi_p(z_{s_i}) \right]^{-1}. \tag{3.10}$$

Theorem 3.7 ([7, Theorem 3.7]). *Let* $F(z)$ *be the vector-valued rational function in* (3.1), *precisely as described in the first paragraph of this section, with the notation therein. With* $u_{i,j}$ *and* $\alpha_{i,s}$ *as in*

[4]Note that the error formula for $F(z) - R_{p,k}(z)$ in case of IMMPE is *precisely of the form* given in (3.3)–(3.5) of Lemma 3.4, but with *different* $\Psi_p(z)$, $u_{i,j}$, and $\Delta_j(z)$; namely, (i) $\Psi_p(z) = \psi_{1,p+1}(z)$, (ii) $u_{i,j} = \alpha_{i,s}\psi_{1,j}(z)/\Psi_p(z)$ with $\alpha_{i,s} = (q_i, v_s)$ and (iii) $\Delta_j(z) = \psi_{1,p}(z) \sum_{s=1}^{\mu} \widehat{e}_s^{(p)}(z)\psi_{1,j}(z_s)/\Psi_p(z_s)$ with $\widehat{e}_s^{(p)}(z) = v_s(z_s - \xi_{p+1})/(z - z_s)$. See [7].

(3.7), and $\widehat{e}_s^{(p)}(z)$ as in (3.8), define

$$\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) = \begin{vmatrix} \widehat{e}^{(p)}_{s_0}(z) & \widehat{e}^{(p)}_{s_1}(z) & \cdots & \widehat{e}^{(p)}_{s_k}(z) \\ \alpha_{1,s_0} & \alpha_{1,s_1} & \cdots & \alpha_{1,s_k} \\ \alpha_{2,s_0} & \alpha_{2,s_1} & \cdots & \alpha_{2,s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,s_0} & \alpha_{k,s_1} & \cdots & \alpha_{k,s_k} \end{vmatrix}. \tag{3.11}$$

Then, with $p > k + \deg(u)$, we have

$$\Delta(z) = (-1)^k \psi_{1,p}(z) \sum_{1\le s_0<s_1<\cdots<s_k\le\mu} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) V(z_{s_0}, z_{s_1}, \ldots, z_{s_k}) \left[\prod_{i=0}^{k} \Psi_p(z_{s_i})\right]^{-1}. \tag{3.12}$$

Finally, combining (3.10) and (3.12) in (3.4), we obtain a simple and elegant expression for $F(z) - R_{p,k}(z)$. This is the subject of the following theorem.

Theorem 3.8 ([7, Theorem 3.8])**.** *For the error in $R_{p,k}(z)$, with $p > k + \deg(u)$, we have the closed-form expression*

$$F(z) - R_{p,k}(z) = \psi_{1,p}(z) \frac{\displaystyle\sum_{1\le s_0<s_1<\cdots<s_k\le\mu} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) V(z_{s_0}, z_{s_1}, \ldots, z_{s_k}) \left[\prod_{i=0}^{k} \Psi_p(z_{s_i})\right]^{-1}}{\displaystyle\sum_{1\le s_1<s_2<\cdots<s_k\le\mu} T_{s_1,s_2,\ldots,s_k} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left[\prod_{i=1}^{k} \Psi_p(z_{s_i})\right]^{-1}}. \tag{3.13}$$

Remark 3.9. When $k = \mu$ in Theorem 3.8, the summation in the numerator on the right-hand side of (3.13) is empty. Thus, this theorem provides an independent proof of the reproducing property of ITEA when $F(z)$ has only simple poles.

§4. Preliminaries for convergence theory

Let E be a closed and bounded set in the z-plane, whose complement K, including the point at infinity, has a classical Green's function $g(z)$ with a pole at infinity, which is continuous on ∂E, the boundary of E, and is zero on ∂E. For each σ, let Γ_σ be the locus $g(z) = \log \sigma$, and let E_σ denote the interior of Γ_σ. Then E_1 is the interior of E and, for $1 < \sigma < \sigma'$, there holds $E \subset E_\sigma \subset E_{\sigma'}$.

For each $p \in \{1, 2, \ldots\}$, let

$$\Xi_p = \{\xi_1^{(p)}, \xi_2^{(p)}, \ldots, \xi_{p+k}^{(p)}\}$$

be the set of interpolation points used in constructing the ITEA interpolant $R_{p,k}(z)$ [5]. Assume that the sets Ξ_p are such that $\xi_i^{(p)}$ have no limit points in K and

$$\lim_{p\to\infty} \left| \prod_{i=1}^{p+k} \left(z - \xi_i^{(p)}\right) \right|^{1/p} = \kappa\Phi(z); \quad \kappa = \mathrm{cap}\,(E), \quad \Phi(z) = \exp[g(z)], \tag{4.1}$$

uniformly in z on every compact subset of K, where cap(E) is the logarithmic capacity of E defined by

$$\mathrm{cap}\,(E) = \lim_{n\to\infty} \Big(\min_{r\in\mathcal{P}_n} \max_{z\in E} |r(z)| \Big)^{1/n}; \quad \mathcal{P}_n = \big\{ r(z) : \; r \in \Pi_n \text{ and monic} \big\}.$$

Such sequences $\{\xi_1^{(p)}, \xi_2^{(p)}, \ldots, \xi_{p+k}^{(p)}\}$, $p = 1, 2, \ldots$, exist (see Walsh [12, p. 74]). Note that, in terms of $\Phi(z)$, the locus Γ_σ is defined by $\Phi(z) = \sigma$ for $\sigma > 1$, while $\partial E = \Gamma_1$ is simply the locus $\Phi(z) = 1$.

Recalling that $\prod_{i=1}^{p+k} \left(z - \xi_i^{(p)}\right) = \Psi_p(z)$ (see (3.6)), we can write (4.1) also as

$$\lim_{p\to\infty} \big|\Psi_p(z)\big|^{1/p} = \kappa\Phi(z), \tag{4.2}$$

uniformly in z on every compact subset of K [6].

It is clear that

$$z' \in \Gamma_{\sigma'}, \quad z'' \in \Gamma_{\sigma''} \quad \text{and} \quad 1 < \sigma' < \sigma'' \quad \Rightarrow \quad 1 < \Phi(z') < \Phi(z''). \tag{4.3}$$

§5. Convergence theory for vector-valued rational $F(z)$ with simple poles

In this section, we provide a convergence theory, in case $F(z)$ is a vector-valued rational function with simple poles as in (3.1), for the sequences $\{R_{p,k}(z)\}_{p=1}^{\infty}$ with $k < \mu$ and fixed[7]. Recall that the sets Ξ_p

[5] Thus, we are now allowing the interpolation points defining $R_{p,k}(z)$ to vary with p.

[6] Note that the definition of $\Phi(z)$ for ITEA given in (4.1) and (4.2) is of *the same form* as the definition of $\Phi(z)$ for IMMPE, but the two differ; for IMMPE, $\lim_{p\to\infty} \big| \prod_{i=1}^{p+1} \left(z - \xi_i^{(p)}\right) \big|^{1/p} = \lim_{p\to\infty} \big|\Psi_p(z)\big|^{1/p} = \kappa\Phi(z)$, where $\kappa = \mathrm{cap}(E)$ as usual.

[7] Note that by the reproducing property mentioned in Section 1, for $k = \mu$, $R_{p,k}(z) = F(z)$ for all $p \geq p_0$, where $p_0 - 1$ is the degree of the numerator of $F(z)$. Thus, there is nothing to prove for the case $k = \mu$.

of points of interpolation that define the approximations $R_{p,k}(z)$ are allowed to vary with p. Also, as we will let $p \to \infty$ in our analysis, the condition that $p > k + \deg(u)$, which is necessary for theorems 3.6, 3.7 and 3.8, is satisfied for all large p.

We continue to use the notation of the preceding sections. We now turn to $F(z)$ in (3.1). We assume that $F(z)$ is analytic in E. This implies that its poles $z_1, \ldots, z_\mu$ are all in K, the complement of E. Now we order the poles of $F(z)$ such that

$$\Phi(z_1) \leq \Phi(z_2) \leq \cdots \leq \Phi(z_\mu). \tag{5.1}$$

By (4.3), if z' and z'' are two different poles of $F(z)$ and $\Phi(z') < \Phi(z'')$, then z' and z'' lie on two different loci $\Gamma_{\sigma'}$ and $\Gamma_{\sigma''}$. In addition, $\sigma' < \sigma''$, that is, the set $E_{\sigma'}$ is in the interior of $E_{\sigma''}$.

5.1. Convergence analysis for $V_{p,k}(z)$

We now state a König-type convergence theorem for $V_{p,k}(z)$ and another theorem concerning its zeros. Since all our results eventually rely on the assumption that $T_{1,2,\ldots,k} \neq 0$, we start by exploring the minimal conditions under which this assumption may hold for ITEA.

Lemma 5.1. *The determinant $T_{s_1,\ldots,s_k}$ defined in* (3.9) *is actually of the form*

$$T_{s_1,\ldots,s_k} = (-1)^{k(k-1)/2} V(z_{s_1}, \ldots, z_{s_k}) \prod_{i=1}^{k} (q, v_{s_i}). \tag{5.2}$$

Proof. Invoking $\alpha_{i,s} = (q, v_s)\psi_{p+i+1,p+k}(z_s)$ (see (3.7)) in (3.9) and letting $\beta_i = (q, v_i)$ for simplicity of notation, we have

$$T_{s_1,\ldots,s_k} = \begin{vmatrix} \beta_{s_1}\psi_{p+2,p+k}(z_{s_1}) & \beta_{s_2}\psi_{p+2,p+k}(z_{s_2}) & \cdots & \beta_{s_k}\psi_{p+2,p+k}(z_{s_k}) \\ \beta_{s_1}\psi_{p+3,p+k}(z_{s_1}) & \beta_{s_2}\psi_{p+3,p+k}(z_{s_2}) & \cdots & \beta_{s_k}\psi_{p+3,p+k}(z_{s_k}) \\ \vdots & \vdots & & \vdots \\ \beta_{s_1}\psi_{p+k+1,p+k}(z_{s_1}) & \beta_{s_2}\psi_{p+k+1,p+k}(z_{s_2}) & \cdots & \beta_{s_k}\psi_{p+k+1,p+k}(z_{s_k}) \end{vmatrix}, \tag{5.3}$$

which, upon factoring out $\beta_{s_1}, \ldots, \beta_{s_k}$, becomes

$$T_{s_1,\ldots,s_k} = T'_{s_1,\ldots,s_k} \prod_{i=1}^{k} \beta_{s_i}, \tag{5.4}$$

where

$$T'_{s_1,\ldots,s_k} = \begin{vmatrix} \psi_{p+2,p+k}(z_{s_1}) & \psi_{p+2,p+k}(z_{s_2}) & \cdots & \psi_{p+2,p+k}(z_{s_k}) \\ \psi_{p+3,p+k}(z_{s_1}) & \psi_{p+3,p+k}(z_{s_2}) & \cdots & \psi_{p+3,p+k}(z_{s_k}) \\ \vdots & \vdots & & \vdots \\ \psi_{p+k+1,p+k}(z_{s_1}) & \psi_{p+k+1,p+k}(z_{s_2}) & \cdots & \psi_{p+k+1,p+k}(z_{s_k}) \end{vmatrix}. \tag{5.5}$$

Now, $\psi_{p+i+1,p+k}(z)$ is a monic polynomial of degree $k-i$, $i = 1,\ldots,k$. Therefore, after permuting the rows of the determinant $T'_{s_1,\ldots,s_k}$ suitably, we can apply Lemma 3.1 and obtain

$$T'_{s_1,\ldots,s_k} = (-1)^{k(k-1)/2} V(z_{s_1},\ldots,z_{s_k}).$$

This completes the proof. ■

Remark 5.2. Judging from (5.3)–(5.5), we may be led to believe that $T_{s_1,\ldots,s_k}$ is actually a function of p. The result in (5.2) shows that it is *independent of* p, and this is quite surprising. In addition, the fact that $T_{s_1,\ldots,s_k}$ is independent of p makes the rest of the proofs possible.

Theorem 5.3 that follows concerns the convergence of $V_{p,k}(z)$ as $p \to \infty$.

Theorem 5.3 ([7, Theorem 5.1])**.** *Assume*

$$\Phi(z_k) < \Phi(z_{k+1}) = \cdots = \Phi(z_{k+r}) < \Phi(z_{k+r+1}), \tag{5.6}$$

in addition to (5.1)*. In case $k + r = \mu$, we define $\Phi(z_{k+r+1}) = \infty$. Assume also that*

$$\prod_{i=1}^{k} (q, v_i) \neq 0.$$

Consequently,

$$T_{1,\ldots,k} \neq 0 \tag{5.7}$$

and there holds

$$Q(z) = (-1)^k T_{1,\ldots,k} V(z, z_1, \ldots, z_k) \left[\prod_{i=1}^{k} \Psi_p(z_i) \right]^{-1} \left[1 + O\left(\frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}} \right) \right] \quad \textit{as } p \to \infty,$$

uniformly in every compact subset of $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_k\}$, *where*

$$\left|\widetilde{\Psi}_{p,k}\right| = \min_{1 \le j \le r} \left|\Psi_p(z_{k+j})\right|. \tag{5.8}$$

Thus, with the normalization that $c_k = 1$, *and letting*

$$S(z) = \prod_{i=1}^{k}(z - z_i),$$

there holds

$$V_{p,k}(z) - S(z) = O\left(\frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \to \infty,$$

from which we also have

$$\limsup_{p\to\infty} \left|V_{p,k}(z) - S(z)\right|^{1/p} \le \frac{\Phi(z_k)}{\Phi(z_{k+1})}.$$

Theorem 5.3 implies that $V_{p,k}(z)$ has precisely k zeros that tend to those of $S(z)$. Let us denote the zeros of $V_{p,k}(z)$ by $z_m^{(p)}$, $m = 1, \ldots, k$. Then $\lim_{p\to\infty} z_m^{(p)} = z_m$, $m = 1, \ldots, k$. In the next theorem, we provide the rate of convergence of each of these zeros.

Theorem 5.4 ([7, Theorem 5.2])**.** *Under the conditions of* Theorem 5.3, *there holds*

$$z_m^{(p)} - z_m = O\left(\frac{\Psi_p(z_m)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \to \infty,$$

with $\widetilde{\Psi}_{p,k}$ *as in* (5.8). *From this, it follows that*

$$\limsup_{p\to\infty} \left|z_m^{(p)} - z_m\right|^{1/p} \le \frac{\Phi(z_m)}{\Phi(z_{k+1})}, \quad m = 1, \ldots, k.$$

In case $r = 1$ *in* (5.6), *that is,*

$$\Phi(z_k) < \Phi(z_{k+1}) < \Phi(z_{k+2}),$$

and assuming that $T_{1,\ldots,m-1,m+1,\ldots,k+1} \neq 0$, *we have the more refined result*

$$z_m^{(p)} - z_m \sim C_m \frac{\Psi_p(z_m)}{\Psi_p(z_{k+1})} \quad \text{as } p \to \infty,$$

$$C_m = (-1)^{k-m} \frac{T_{1,\dots,m-1,m+1,\dots,k+1}}{T_{1,\dots,k}} (z_{k+1} - z_m) \prod_{\substack{i=1 \\ i \neq m}}^{k} \frac{z_{k+1} - z_i}{z_m - z_i}.$$

5.2. Convergence analysis for $R_{p,k}(z)$

We now develop a de Montessus type convergence theory for the $R_{p,k}(z)$; that is, we analyze the error $F(z) - R_{p,k}(z)$ as $p \to \infty$ with k being held fixed.

We start by showing that the vectors $\widehat{T}^{(p)}_{s_0,s_1,\dots,s_k}(z)$ are (i) meromorphic in z with simple poles at the z_i and (ii) bounded for all large p. This is the subject of the lemma that follows.

Lemma 5.5. *For $z \notin \{z_{s_0}, z_{s_1} \dots, z_{s_k}\}$, $\widehat{T}^{(p)}_{s_0,s_1,\dots,s_k}(z)$ is analytic in z and bounded for all large p.*

Proof. Expanding the vector-valued determinant in (3.11) with respect to its first row, we obtain

$$\widehat{T}^{(p)}_{s_0,s_1,\dots,s_k}(z) = \sum_{j=0}^{k} E_j \widehat{e}^{(p)}_{s_j}(z),$$

where

$$E_j = (-1)^j T_{s_0,\dots,s_{j-1},s_{j+1},\dots,s_k}, \quad \widehat{e}^{(p)}_{s_j}(z) = \frac{v_{s_j}}{z - z_{s_j}} \prod_{i=p+1}^{p+k} (z_{s_j} - \xi^{(p)}_i), \quad j = 0, 1, \dots, k.$$

By Lemma 5.1, E_j are all scalars independent of p. In addition, $\widehat{e}^{(p)}_{s_j}(z)$ are bounded in p since $\xi^{(p)}_{p+1}, \dots, \xi^{(p)}_{p+k}$ are bounded due to the assumption that the $\xi^{(p)}_i$ have no limit points in K, and k is a fixed integer. This completes the proof. ■

We make use of Lemma 5.5 in the proof of Theorem 5.6 that follows. Throughout the rest of this work, $\|Y\|$ denotes the vector norm of $Y \in \mathbb{C}^N$.

Theorem 5.6 ([7, Theorem 5.3]). *Under the conditions of* Theorem 5.3, *$R_{p,k}(z)$ exists and is unique and satisfies*

$$F(z) - R_{p,k}(z) = O\left(\frac{\Psi_p(z)}{\widetilde{\Psi}_{p,k}}\right) \quad \textit{as } p \to \infty,$$

uniformly on every compact subset of $\mathbb{C}\setminus\{z_1,\ldots,z_\mu\}$, *with* $\widetilde{\Psi}_{p,k}$ *as defined in* (5.8). *From this, it also follows that*

$$\limsup_{p\to\infty}\left\|F(z)-R_{p,k}(z)\right\|^{1/p}\leq\frac{\Phi(z)}{\Phi(z_{k+1})},\quad z\in\widetilde{K}=K\setminus\{z_1,\ldots,z_\mu\},$$

uniformly on each compact subset of $\widetilde{K}$, *and*

$$\limsup_{p\to\infty}\left\|F(z)-R_{p,k}(z)\right\|^{1/p}\leq\frac{1}{\Phi(z_{k+1})},\quad z\in E,$$

uniformly on E. *Thus, uniform convergence takes place for* z *in any compact subset of the set* $\widetilde{K}_k$, *where*

$$\widetilde{K}_k=\operatorname{int}\Gamma_{\sigma_k}\setminus\{z_1,\ldots,z_k\};\quad \sigma_k=\Phi(z_{k+1}).$$

When $r=1$ *in* (5.6), *that is, when*

$$\Phi(z_k)<\Phi(z_{k+1})<\Phi(z_{k+2}),$$

and $\widehat{T}^{(p)}_{1,\ldots,k+1}(z)\neq 0$ *in addition to* (5.7), *we have the more refined result*

$$F(z)-R_{p,k}(z)\sim B_p(z)\frac{\psi_{1,p}(z)}{\Psi_p(z_{k+1})}\quad \text{as } p\to\infty,\quad B_p(z)=(-1)^k\frac{\widehat{T}^{(p)}_{1,\ldots,k+1}(z)}{T_{1,\ldots,k}}\prod_{i=1}^{k}\frac{z_{k+1}-z_i}{z-z_i},$$

and $B_p(z)$ *is bounded for all large* p.

§6. Convergence theory for general vector-valued meromorphic $F(z)$ with simple poles

Let the sets of interpolation points $\{\xi^{(p)}_1,\ldots,\xi^{(p)}_{p+k}\}$ be as in the preceding section. We now turn to the convergence analysis of $R_{p,k}(z)$ as $p\to\infty$, when the function $F(z)$ is analytic in E and meromorphic in $E_\rho=\operatorname{int}\Gamma_\rho$, where Γ_ρ, as before, is the locus $\Phi(z)=\rho$ for some $\rho>1$. Assume that $F(z)$ has μ simple poles $z_1,\ldots,z_\mu$ in E_ρ. Thus, $F(z)$ has the following form:

$$F(z)=\sum_{s=1}^{\mu}\frac{v_s}{z-z_s}+\Theta(z),\tag{6.1}$$

$\Theta(z)$ being analytic in E_ρ.

The treatment of this case is based entirely on that of the preceding section, the differences being minor. Note that the polynomial $u(z)$ of (3.1) is now replaced by $\Theta(z)$ in (6.1). Previously, we had $u[\xi_m, \ldots, \xi_n] = 0$ for all large $n - m$, as a consequence of which, we had (3.7) for $u_{i,j}$ and (3.8) for $\Delta_j(z)$. Instead of these, we now have

$$u_{i,j} = -\sum_{s=1}^{\mu} \alpha_{i,s} \frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)} + \big(q, \Theta[\xi_{j+1}, \ldots, \xi_{p+i}]\big), \tag{6.2}$$

with $\alpha_{i,s}$ as in (3.7), and

$$\Delta_j(z) = \psi_{1,p}(z) \Bigg(\sum_{s=1}^{\mu} \widehat{e}_s^{(p)}(z) \frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)} + \Theta[z, \xi_{j+1}, \ldots, \xi_p] \Bigg), \tag{6.3}$$

with $\widehat{e}_s^{(p)}(z)$ as in (3.8).

It is clear that the treatment of the general meromorphic $F(z)$ with simple poles will be the same as that of the rational $F(z)$ with simple poles provided the contributions from $\Theta(z)$ to $u_{i,j}$ and $\Delta_j(z)$, as $p \to \infty$, are negligible compared to the rest of the terms in (6.2) and (6.3). This is indeed the case, as is shown in [7, Lemma 6.1]:

Lemma 6.1 ([7, Lemma 6.1]). *With $F(z)$ as in the first paragraph, there holds*

$$\limsup_{p\to\infty} \big\|\Theta[\xi_{j+1}^{(p)}, \ldots, \xi_{p+i}^{(p)}]\big\|^{1/p} \le \frac{1}{\kappa\rho}.$$

There also holds

$$\limsup_{p\to\infty} \big\|\Theta[z, \xi_{j+1}^{(p)}, \ldots, \xi_p^{(p)}]\big\|^{1/p} \le \frac{1}{\kappa\rho},$$

uniformly in every compact subset of E_ρ. These hold for all $i \le k$ and $j \le k$.

With this information, we can now prove convergence results for $V_{n,k}(z)$ and $F(z) - R_{p,k}(z)$ for general meromorphic $F(z)$. We recall that the poles $z_1, \ldots, z_\mu$ of $F(z)$ are ordered such that

$$\Phi(z_1) \le \Phi(z_2) \le \cdots \le \Phi(z_\mu) \le \rho. \tag{6.4}$$

We also adopt the notation of theorems 5.3, 5.4 and 5.6.

Theorem 6.2 ([7, Theorem 6.2])**.**

(i) *When* $k < \mu$*, assume that*

$$\Phi(z_k) < \Phi(z_{k+1}) = \cdots = \Phi(z_{k+r}) < \begin{cases} \Phi(z_{k+r+1}), & \textit{if } k+r < \mu, \\ \rho, & \textit{if } k+r = \mu, \end{cases}$$

in addition to (6.4)*. Assume also that*

$$\prod_{i=1}^{k} (q, v_i) \neq 0.$$

Consequently,

$$T_{1,\ldots,k} \neq 0.$$

Then, all the results of Theorem 5.3 *hold.*

(ii) *When* $k = \mu$*,*

$$\limsup_{p\to\infty} \left|V_{p,k}(z) - S(z)\right|^{1/p} \leq \frac{\Phi(z_k)}{\rho},$$

uniformly on every compact subset of $\mathbb{C} \setminus \{z_1, \ldots, z_\mu\}$*.*

Theorem 6.2 implies that $V_{p,k}(z)$ has precisely k zeros that tend to those of $S(z)$. Let us denote the zeros of $V_{p,k}(z)$ by $z_m^{(p)}$, $m = 1, \ldots, k$. Then $\lim_{p\to\infty} z_m^{(p)} = z_m$, $m = 1, \ldots, k$. In the next theorem, we provide the rate of convergence of each of these zeros.

Theorem 6.3 ([7, Theorem 6.3])**.** *Assume the conditions of* Theorem 5.4.

(i) *When* $k < \mu$*, all the results of* Theorem 5.4 *hold.*

(ii) *When* $k = \mu$*,*

$$\limsup_{p\to\infty} \left|z_m^{(p)} - z_m\right|^{1/p} \leq \frac{\Phi(z_m)}{\rho}, \quad m = 1, \ldots, k.$$

Our next and last result concerns the convergence of $R_{p,k}(z)$.

Theorem 6.4 ([7, Theorem 6.4]). *Assume the conditions of* Theorem 5.6. *Then* $R_{p,k}(z)$ *exists and is unique.*

(i) *When* $k < \mu$, *all the results of* Theorem 5.6 *hold with* $\widetilde{K} = E_\rho \setminus \{z_1, \ldots, z_\mu\}$.

(ii) *When* $k = \mu$, *there holds*

$$\limsup_{p\to\infty} \left\|F(z) - R_{p,k}(z)\right\|^{1/p} \leq \frac{\Phi(z)}{\rho}, \quad z \in \widetilde{K} = E_\rho \setminus \{z_1, \ldots, z_\mu\},$$

uniformly on each compact subset of $\widetilde{K}$, *and*

$$\limsup_{p\to\infty} \left\|F(z) - R_{p,k}(z)\right\|^{1/p} \leq \frac{1}{\rho}, \quad z \in E,$$

uniformly on E.

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