



Recent Developments in Asymptotic Expansions From Numerical Analysis and Approximation Theory

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Abstract

In this chapter, we discuss some recently obtained asymptotic expansions related to problems in numerical analysis and approximation theory.

- We present a generalization of the Euler–Maclaurin (E–M) expansion for the trapezoidal rule approximation of finite-range integrals $\int_a^b f(x)dx$, when $f(x)$ is allowed to have arbitrary algebraic–logarithmic endpoint singularities. We also discuss effective numerical quadrature formulas for so-called *weakly singular*, *singular*, and *hypersingular* integrals, which arise in different problems of applied mathematics and engineering.
- We present a full asymptotic expansion (as the number of abscissas tends to infinity) for Gauss–Legendre quadrature for finite-range integrals $\int_a^b f(x)dx$, where $f(x)$ is allowed to have arbitrary algebraic–logarithmic endpoint singularities.

- We present full asymptotic expansions, as $n \rightarrow \infty$, (i) for Legendre polynomials $P_n(x)$, $x \in (-1, 1)$, (ii) for the integral $\int_c^d f(x)P_n(x)dx$, $-1 < c < d < 1$, and (iii) for Legendre series coefficients $e_n[f] = (n+1/2) \int_{-1}^1 f(x)P_n(x)dx$, when $f(x)$ has arbitrary algebraic–logarithmic (interior and/or endpoint) singularities in $[-1, 1]$.



1. INTRODUCTION

In many problems of science and engineering, one is confronted with the problem of determining the asymptotic behavior of some function $f(x)$ as $x \rightarrow a$ for some fixed a ; typically, $a = 0$ or $a = \infty$.

1. In some cases, the best one can do is to obtain an *upper bound* for $|f(x)|$ as $x \rightarrow a$; that is, one can have

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow a, \quad g(x) \text{ a known simple function.} \quad (1)$$

Of course, this means that there exist fixed positive constants M , ϵ , and X , for which,

$$|f(x)| \leq M |g(x)| \quad \begin{cases} \forall x \in (a - \epsilon, a + \epsilon) \text{ or } \forall x \in (a, a + \epsilon) \text{ if } a \text{ finite,} \\ \forall x \in (X, \infty) \text{ if } a = \infty. \end{cases} \quad (2)$$

2. In other problems, one can obtain the *actual* asymptotic behavior of $f(x)$ as $x \rightarrow a$ in the form of an *asymptotic equality*, namely,

$$f(x) \sim h(x) \quad \text{as } x \rightarrow a, \quad h(x) \text{ a known simple function,} \quad (3)$$

which means that

$$\lim_{x \rightarrow a} \frac{f(x)}{h(x)} = 1. \quad (4)$$

3. Yet in other cases, one can obtain a *complete asymptotic expansion* for $f(x)$ as $x \rightarrow a$ that is in the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x) \quad \text{as } x \rightarrow a, \quad (5)$$

where c_n are some constants and the functions $\phi_n(x)$ are simple known functions such that the sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ is an *asymptotic scale*, namely,

$$\lim_{x \rightarrow a} \frac{\phi_{n+1}(x)}{\phi_n(x)} = 0, \quad n = 0, 1, \dots \quad (6)$$

(For examples, the functions $\phi_n(x) = (x - a)^n$, with a finite, form an asymptotic scale as $x \rightarrow a$. Similarly, the functions $\phi_n(x) = x^{-n}$ form an asymptotic scale as $x \rightarrow \infty$).

By (5), we mean

$$f(x) - \sum_{n=0}^{N-1} c_n \phi_n(x) = O(\phi_N(x)) \quad \text{as } x \rightarrow a. \quad (7)$$

In this case, if $c_n = 0$ for $0 \leq n \leq k - 1$ and $c_k \neq 0$, we also have the asymptotic equality

$$f(x) \sim c_k \phi_k(x) \quad \text{as } x \rightarrow a. \quad (8)$$

If $c_n = 0$ for all n , then we have that $\lim_{x \rightarrow a} f(x)/\phi_n(x) = 0$ for all n . (Note that the infinite series $\sum_{n=0}^{\infty} c_n \phi_n(x)$ may be convergent or divergent and this does not present an issue we need to worry about.)

Clearly, the situation described in item 2 is more informative than that described in item 1 since it implies the latter, and the situation described in item 3 is more informative than that described in item 2 and implies the latter.

Our purpose in this work is to give a review of some recently obtained *complete* asymptotic expansions for some commonly occurring problems arising in numerical analysis and approximation theory that have also been observed to occur in other disciplines, such as applied mathematics and theoretical physics, for example.

In Section 2, we discuss the error expansion for trapezoidal rule approximations of finite-range integrals $\int_a^b f(x) dx$ (as the number of abscissas tends to infinity) when $f(x)$ is allowed to have arbitrary algebraic–logarithmic endpoint singularities in general, and present recent generalizations of the well-known Euler–Maclaurin (E–M) expansions. We also present very effective numerical quadrature formulas for some singular integrals that are derived from these expansions.

In Section 3, we present the error expansion of Gauss–Legendre quadrature formulas for finite-range integrals $\int_a^b f(x) dx$ (again, as the number of abscissas tends to infinity), where $f(x)$ is allowed to have arbitrary algebraic–logarithmic endpoint singularities.

In Section 4, we discuss the asymptotic expansions as $n \rightarrow \infty$ (i) for Legendre polynomials $P_n(x)$, $x \in (-1, 1)$, (ii) for the integral $\int_c^d f(x) P_n(x) dx$, $-1 < c$

$< d < 1$, and (iii) for the coefficients $e_n[f] = (n + 1/2) \int_{-1}^1 f(x) P_n(x) dx$ of the Legendre series $\sum_{n=0}^{\infty} e_n[f] P_n(x)$ of a function $f(x)$, when $f(x)$ has arbitrary algebraic–logarithmic (interior and/or endpoint) singularities in $[-1, 1]$. These asymptotic expansions, in addition to being of interest by themselves, can have applications in asymptotic analyses involving Legendre expansions, such as integral equations, numerical quadrature, and in series of spherical harmonics.

Before closing, we mention that, when computing the integrals $\int_a^b f(x) dx$, where $f(x)$ has singularities at the endpoints $x = a$ and $x = b$, we can first use suitable variable transformations and apply the trapezoidal rule or the Gauss–Legendre quadrature to the transformed integral; by this, we can achieve very high accuracy. When $f(x)$ has asymptotic expansions of the types discussed in this work, some of the variable transformation can be tuned to enable the quadrature formulas to attain accuracies that are optimal in some asymptotic sense. We do not treat this subject here; we refer the reader to Sidi¹ and the references therein.

Finally, for simplicity of notation, in the sequel, we will write “ $(h \rightarrow 0)$ ” or the equivalent “ $(n \rightarrow \infty)$ ” instead of “as $h \rightarrow 0$ ” or the equivalent “as $n \rightarrow \infty$ ”.



2. TRAPEZOIDAL RULE APPROXIMATIONS AND GENERALIZATIONS OF THE E–M EXPANSION

2.1 Classical E–M Expansion

We start with the well-known classical E–M expansion for finite-range integrals $I[f] = \int_a^b f(x) dx$, where $f \in C^\infty[a, b]$, which involves the trapezoidal rule approximation for $I[f]$. We summarize the subject in the following theorem.

Theorem 1. *Let*

$$I[f] = \int_a^b f(x) dx \text{ and } T_n[f] = h \left[\frac{1}{2} f(a) + \sum_{i=1}^{n-1} f(a+ih) + \frac{1}{2} f(b) \right], \quad h = \frac{b-a}{n}. \quad (9)$$

Assume that $f \in C^\infty[a, b]$. Then there holds

$$T_n[f] \sim I[f] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] h^{2k} \quad (h \rightarrow 0). \quad (10)$$

Here B_s are the Bernoulli numbers.

The following are immediate consequences of [Theorem 1](#):

$$f'(a) \neq f'(b) \Rightarrow T_n[f] - I[f] \sim \frac{1}{12}[f'(b) - f'(a)]h^2 = O(h^2) \quad (h \rightarrow 0).$$

$$f^{(2k-1)}(a) = f^{(2k-1)}(b), \quad 1 \leq k \leq m-1 \Rightarrow T_n[f] - I[f] = O(h^{2m}) \quad (h \rightarrow 0).$$

$$f^{(2k-1)}(a) = f^{(2k-1)}(b), \quad k = 1, 2, \dots \Rightarrow T_n[f] - I[f] = O(h^\mu) \quad (h \rightarrow 0), \quad \forall \mu > 0,$$

and this happens when $f \in C^\infty(-\infty, \infty)$ and is $(b - a)$ -periodic, for example.

The integral $\int_a^b f(x) dx$ can be computed to high accuracy by applying the *Richardson extrapolation process* to the sequence of trapezoidal rule approximations $\{T_{2^n}[f]\}_{n=0}^\infty$ also taking into account the E–M expansion; this method is known as the *Romberg integration*. All this is treated in many numerical analysis books; see Atkinson,² Ralston and Rabinowitz,³ and Stoer and Bulirsch,⁴ for example. See also the book by Steffensen.⁵ For a detailed treatment of extrapolation methods and their applications, see Sidi.⁶

2.2 Navot's Generalizations of Classical E–M Expansion

The first generalizations of the E–M expansion were given by Navot,^{7,8} and these concern the cases in which $f(x)$ has endpoint singularities of algebraic and algebraic–logarithmic types, respectively. Navot's results were later rederived by Lyness and Ninham⁹ using techniques involving generalized functions. We state the result pertaining to the algebraic case in the next theorem. We will deal with the algebraic–logarithmic case when discussing our recent generalizations of the E–M expansions later in this section.

Theorem 2. *Let $f(x)$ have algebraic endpoint singularities as in*

$$f(x) = (x - a)^\alpha g_a(x) = (b - x)^\beta g_b(x), \quad g_a \in C^\infty[a, b), \quad g_b \in C^\infty(a, b], \quad (11)$$

and $\Re\alpha, \Re\beta > -1$ and α, β not necessarily integers. Let also

$$I[f] = \int_a^b f(x) dx \quad \text{and} \quad \check{T}_n[f] = h \sum_{i=1}^{n-1} f(a + ih), \quad h = \frac{b-a}{n}. \quad (12)$$

(Note that $\check{T}_n[f]$ does not include $f(a)$ and $f(b)$; thus it is always defined.)

Then there holds

$$\begin{aligned} \check{T}_n[f] \sim I[f] + \sum_{k=0}^{\infty} \frac{\zeta(-\alpha-k)}{k!} g_a^{(k)}(a) h^{\alpha+k+1} \\ + \sum_{k=0}^{\infty} (-1)^k \frac{\zeta(-\beta-k)}{k!} g_b^{(k)}(b) h^{\beta+k+1} \quad (h \rightarrow 0). \end{aligned} \quad (13)$$

Here $\zeta(\omega)$ is the Riemann zeta function.^a

Note that in case $\alpha = 0 = \beta$, we have $g_a(x) = f(x) = g_b(x)$, and with the help of the known results concerning the zeta function, namely,

$$\zeta(0) = -\frac{1}{2}; \quad \zeta(-2m) = 0, \quad \zeta(1-2m) = -\frac{B_{2m}}{2m}, \quad m = 1, 2, \dots, \quad (14)$$

we recover the (classical) E–M expansion of [Theorem 1](#).

2.3 Recent Generalizations of E–M Expansions

In a recent series of papers by Sidi,^{10–12} the E–M expansion has been extended to the most general case in which the integrand $f(x)$ is infinitely differentiable in the finite (open) interval (a, b) and is allowed to have arbitrary singular behavior of the algebraic–logarithmic types at the endpoints. These results, which we summarize as [Theorem 3](#), contain all earlier ones as special cases, but are not contained in the latter. Before we state them, we mention that they are valid also when the integrals $\int_a^b f(x) dx$ are divergent hence do not exist in the regular sense, but they are defined in the sense of *Hadamard finite part* (HPF). In such cases, we continue to use the notation $I[f]$ to denote the HFP of the (divergent) integral $\int_a^b f(x) dx$. Of course, the HFP of a convergent integral is equal to the actual value of the integral.

Theorem 3. Assume that $f \in C^\infty(a, b)$ and let

$$\check{T}_n[f] = h \sum_{i=1}^{n-1} f(a+ih), \quad h = \frac{b-a}{n}. \quad (15)$$

^a We recall that the Riemann zeta function $\zeta(\omega)$ is first defined via the convergent series $\zeta(\omega) = \sum_{k=1}^{\infty} k^{-\omega}$ for $\Re\omega > 1$, and then continued analytically to the ω -plane. It is analytic everywhere except at $\omega = 1$, where it has a simple pole with residue 1.

We consider the following two cases:

1. Let $f(x)$ have asymptotic expansions as $x \rightarrow a+$ and as $x \rightarrow b-$ as in

$$\begin{aligned} f(x) &\sim K(x-a)^{-1} + \sum_{s=0}^{\infty} c_s(x-a)^{\gamma_s} \quad (x \rightarrow a+) \\ f(x) &\sim L(b-x)^{-1} + \sum_{s=0}^{\infty} d_s(b-x)^{\delta_s} \quad (x \rightarrow b-) \end{aligned} \quad (16)$$

where γ_s and δ_s are in general complex constants satisfying

$$\begin{aligned} \Re\gamma_0 \leq \Re\gamma_1 \leq \Re\gamma_2 \leq \dots; \quad \gamma_s \neq -1, \quad \lim_{s \rightarrow \infty} \Re\gamma_s = \infty, \\ \Re\delta_0 \leq \Re\delta_1 \leq \Re\delta_2 \leq \dots; \quad \delta_s \neq -1, \quad \lim_{s \rightarrow \infty} \Re\delta_s = \infty. \end{aligned} \quad (17)$$

Assume also that the asymptotic expansions in (16) are differentiable infinitely many times. Then

$$\begin{aligned} \check{T}_n[f] &\sim I[f] + K(C - \log h) + \sum_{\substack{s=0 \\ \gamma_s \notin \{2, 4, \dots\}}}^{\infty} c_s \zeta(-\gamma_s) h^{\gamma_s + 1} \\ &\quad + L(C - \log h) + \sum_{\substack{s=0 \\ \delta_s \notin \{2, 4, \dots\}}}^{\infty} d_s \zeta(-\delta_s) h^{\delta_s + 1} \quad (h \rightarrow 0). \end{aligned} \quad (18)$$

Here $C = 0.577\dots$ is Euler's constant.

2. More generally, let $f(x)$ have asymptotic expansions as $x \rightarrow a+$ and as $x \rightarrow b-$ as in

$$\begin{aligned} f(x) &\sim \hat{P}(\log(x-a))(x-a)^{-1} + \sum_{s=0}^{\infty} P_s(\log(x-a))(x-a)^{\gamma_s} \quad (x \rightarrow a+) \\ f(x) &\sim \hat{Q}(\log(b-x))(b-x)^{-1} + \sum_{s=0}^{\infty} Q_s(\log(b-x))(b-x)^{\delta_s} \quad (x \rightarrow b-), \end{aligned} \quad (19)$$

where $\hat{P}(\gamma)$, $P_s(\gamma)$, $\hat{Q}(\gamma)$, and $Q_s(\gamma)$ are arbitrary polynomials in γ , and γ_s and δ_s are in general complex satisfying

$$\begin{aligned} \Re\gamma_0 \leq \Re\gamma_1 \leq \Re\gamma_2 \leq \dots; \quad \gamma_s \neq -1, \quad \lim_{s \rightarrow \infty} \Re\gamma_s = \infty, \\ \Re\delta_0 \leq \Re\delta_1 \leq \Re\delta_2 \leq \dots; \quad \delta_s \neq -1, \quad \lim_{s \rightarrow \infty} \Re\delta_s = \infty, \end{aligned} \quad (20)$$

such that

$$\begin{aligned}\Re\gamma_{s+1} = \Re\gamma_s &\Rightarrow \deg P_{s+1} \leq \deg P_s \text{ and} \\ \Re\delta_{s+1} = \Re\delta_s &\Rightarrow \deg Q_{s+1} \leq \deg Q_s.\end{aligned}\quad (21)$$

Assume also that the asymptotic expansions in (19) are differentiable infinitely many times. Let

$$D_\omega \equiv d/d\omega; \quad \hat{P}(\gamma) = \sum_{i=0}^{\hat{p}} \hat{c}_i \gamma^i, \quad \hat{Q}(\delta) = \sum_{i=0}^{\hat{q}} \hat{d}_i \delta^i. \quad (22)$$

Then

$$\begin{aligned}\check{T}_n[f] \sim I[f] &+ \sum_{i=0}^{\hat{p}} \left[\sum_{r=i}^{\hat{p}} \binom{\hat{p}}{r} \hat{c}_r \sigma_{r-i} \right] (\log h)^i - \sum_{i=0}^{\hat{p}} \hat{c}_i \frac{(\log h)^{i+1}}{i+1} \\ &+ \sum_{s=0}^{\infty} P_s(D_{\gamma_s}) [\zeta(-\gamma_s) h^{\gamma_s+1}] + \sum_{s=0}^{\infty} Q_s(D_{\delta_s}) [\zeta(-\delta_s) h^{\delta_s+1}] \\ &+ \sum_{i=0}^{\hat{q}} \left[\sum_{r=i}^{\hat{q}} \binom{\hat{q}}{r} \hat{d}_r \sigma_{r-i} \right] (\log h)^i - \sum_{i=0}^{\hat{q}} \hat{d}_i \frac{(\log h)^{i+1}}{i+1} \quad (h \rightarrow 0).\end{aligned}\quad (23)$$

Here, σ_i are Stieltjes numbers defined via

$$\sigma_i = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{(\log k)^i}{k} - \frac{(\log n)^{i+1}}{i+1} \right], \quad i = 0, 1, \dots$$

Remark

1. By the asymptotic expansions in (16) and (19) being differentiable infinitely many times we mean that, for each integer $k = 1, 2, \dots, f^{(k)}(x)$, the k -th derivative of $f(x)$, has asymptotic expansions as $x \rightarrow a+$ and as $x \rightarrow b-$ that are obtained by differentiating those in (16) and (19) formally term by term.
2. The function $f(x)$ in [Theorem 1](#) is contained in part 1 of [Theorem 3](#) as follows: Being in $C^\infty[a, b]$, $f(x)$ has Taylor series expansions at $x = a$ and $x = b$ and we have $\gamma_s = \delta_s = s$ and $c_s = f^{(s)}(a)/s!$ and $d_s = (-1)^s f^{(s)}(b)/s!$, $s = 0, 1, \dots$. In addition, these Taylor series are differentiable infinitely many times.^b

^b If $f \in C^\infty[a - \epsilon, a + \epsilon]$ for some $\epsilon > 0$, then its Taylor series $\sum_{s=0}^{\infty} \frac{f^{(s)}(a)}{s!} (x-a)^s$ is differentiable infinitely many times whether it converges or diverges.

3. Similarly, the function $f(x)$ in [Theorem 2](#) is contained in part 1 of [Theorem 3](#) as follows: $\gamma_s = s + \alpha$, $\delta_s = s + \beta$, and $c_s = g_a^{(s)}(a)/s!$, $d_s = (-1)^s g_b^{(s)}(b)/s!$, $s = 0, 1, \dots$
4. The function $f(x) = g(x)(x-a)^\alpha \log(x-a)$ with $g \in C^\infty[a, b]$ (considered in [Ref. 8](#)) is contained in part 2 of [Theorem 3](#) as follows: $\gamma_s = s + \alpha$, $\delta_s = s$ and $P_s(y) = [g^{(s)}(a)/s!]y$, $Q_s(y) = (-1)^s f^{(s)}(b)/s!$, $s = 0, 1, \dots$
5. The singular terms $K(x-a)^{-1}$ and $L(b-x)^{-1}$ in [\(16\)](#) and $\hat{P}(\log(x-a))(x-a)^{-1}$ and $\hat{Q}(\log(b-x))(b-x)^{-1}$ in [\(19\)](#) are not present in any of the earlier generalizations of the E–M expansion. (These were treated in [Refs. 11 and 12](#) for the first time.)
6. $I[f]$ is the exact value of the integral $\int_a^b f(x) dx$ when this converges; it is the HFP of $\int_a^b f(x) dx$ otherwise. Here are a few examples relevant to us:

$$u(x) = (x-a)^\omega, \quad \omega \neq -1 \Rightarrow I[u] = \frac{(b-a)^{\omega+1}}{\omega+1},$$

$$u(x) = (x-a)^{-1} \Rightarrow I[u] = \log(b-a),$$

$$u(x) = (\log(x-a))^i (x-a)^\omega, \quad \omega \neq -1 \Rightarrow I[u] = \frac{d^i (b-a)^{\omega+1}}{d\omega^i \omega + 1},$$

$$u(x) = (\log(x-a))^i (x-a)^{-1} \Rightarrow I[u] = \frac{[\log(b-a)]^{i+1}}{i+1}.$$

$I[u]$ is the exact integral $\int_a^b u(x) dx$ if $\Re\omega > -1$, it is the HFP of $\int_a^b u(x) dx$ otherwise.

7. If $R(y) = \sum_{i=0}^r e_i y^i$ and $D_\omega \equiv d/d\omega$, then $R(D_\omega) = \sum_{i=0}^r e_i D_\omega^i$ is a linear differential operator of order r ; therefore, for any function g that is sufficiently differentiable as a function of ω , we have $R(D_\omega)g \equiv \sum_{i=0}^r e_i (D_\omega^i g) = \sum_{i=0}^r e_i (d^i g/d\omega^i)$. Thus

$$R(D_\omega)[\zeta(-\omega)h^{\omega+1}] = h^{\omega+1} W(\log h), \quad W(y) \text{ polynomial of degree } r \text{ in } y.$$

As a result, the term $P_s(D_{\gamma_s})[\zeta(-\gamma_s)h^{\gamma_s+1}]$ in [\(23\)](#) is simply the product of h^{γ_s+1} and a polynomial in $\log h$ whose degree is precisely $\deg P_s$. Similarly, for $Q_s(D_{\delta_s})[\zeta(-\delta_s)h^{\delta_s+1}]$.

8. Note that with $f(x)$ as in part 1 of [Theorem 3](#) and $K = L = 0$, when $\gamma_0, \delta_0 \notin \mathbb{Z}^+$, we have at worst $\check{T}_n[f] - I[f] = O(n^{-(\alpha+1)})$ as $n \rightarrow \infty$, where $\alpha = \min\{\Re\gamma_0, \Re\delta_0\}$.

We would like to emphasize that these remarks about [Theorem 3](#) are relevant to [Theorems 6 and 9](#) that we state in the sequel.

2.4 Applications to Computation of Singular Integrals

E–M expansions and their generalizations have been very useful in the development of numerical quadrature formulas of high accuracy for numerical computation of integrals $\int_a^b f(x) dx$, where $f(x)$ has a singularity at $x = t \in (a, b)$, which may be algebraic and/or logarithmic, such that $f(x)$ may or may not be integrable through $x = t$. Commonly occurring cases are those for which

$$f(x) = g(x) \log|x - t| + \tilde{g}(x), \quad (24)$$

$$f(x) = \frac{g(x)}{(x - t)^m}, \quad m = 1, 2, \dots, \quad (25)$$

$$f(x) = \frac{g(x)}{|x - t|^\beta}, \quad \beta \geq 1 \text{ arbitrary}, \quad (26)$$

$g(x)$ and $\tilde{g}(x)$ being well behaved in (a, b) . The first case involving $\log|x - t|$ arises in so-called *weakly singular* Fredholm integral equations; the integral $\int_a^b f(x) dx$ exists in the regular sense in this case. The second case with $m = 1$ and $m = 2$ arises in so-called *singular* and *hypersingular* Fredholm integral equations: (i) when $m = 1$, $\int_a^b f(x) dx$ diverges but exists as a *Cauchy principal value* integral, and (ii) when $m = 2$, $\int_a^b f(x) dx$ diverges but exists as an *HPF* integral and is also called a *hypersingular* integral.

By manipulating the relevant generalized E–M expansions in suitable ways, numerical quadrature formulas for all these cases were developed and analyzed in the papers by Sidi and Israeli¹³ and Sidi.^{14–17} We do not go into the construction of these formulas here. We mention only the methods that were designed for functions $f(x)$ that are *periodic with period* $T = b - a$ and are infinitely differentiable for all x except $x = t \pm kT$, $k = 0, 1, 2, \dots$. Letting $h = (b - a)/n$, we define the numerical quadrature formulas $Q_n[f]$ as follows:

- For $f(x)$ as in (24),

$$Q_n[f] = h \sum_{j=1}^{n-1} f(t + jh) + \tilde{g}(t)h + g(t)h \log\left(\frac{h}{2\pi}\right),$$

and we have the asymptotic expansion

$$Q_n[f] \sim I[f] + \sum_{k=1}^{\infty} w_k h^{2k+1} \quad (h \rightarrow 0); \quad w_k = -2 \frac{\zeta'(-2k)}{(2k)!} g^{(2k)}(t).$$

Thus, $Q_n[f] - I[f] \sim w_1 h^3$ as $h \rightarrow 0$. We can now apply the Richardson extrapolation process to a sequence $\{Q_{n_i}[f]\}_{i=0}^{\infty}$ with $n_i = n_0 2^i$, $i = 0, 1, \dots$, and some arbitrary n_0 , for example, and obtain highly accurate approximations to $I[f]$.

- For $f(x)$ as in (25) with $m = 1$,

$$Q_n[f] = h \sum_{j=1}^n f(t + jh - h/2),$$

and we have

$$Q_n[f] - I[f] = O(h^\mu) \quad (h \rightarrow 0), \quad \forall \mu > 0,$$

which means that the error tends to zero faster than every positive power of h .

- For $f(x)$ as in (25) with $m = 2$,

$$Q_n[f] = h \sum_{j=1}^n f(t + jh - h/2) - \pi^2 g(t) h^{-1},$$

and we have

$$Q_n[f] - I[f] = O(h^\mu) \quad (h \rightarrow 0), \quad \forall \mu > 0,$$

which means that the error tends to zero faster than every positive power of h .

All three numerical quadrature formulas are used in the numerical solution of Fredholm integral equations mentioned above.

2.5 Further Remarks on E–M Expansion and Generalizations

1. The asymptotic expansions described in [Theorems 1–3](#) concern the *trapezoidal rule* approximations $T_n[f]$ and $\tilde{T}_n[f]$ for $\int_a^b f(x) dx$. Similar but more general versions of these theorems that concern the so-called *offset trapezoidal rule* approximations that are defined as in

$$\tilde{T}_n[f] = h \sum_{j=0}^{n-1} f(t + jh + \theta h), \quad h = \frac{b-a}{n}, \quad \theta \in [0, 1],$$

have been considered in Refs. [5](#) (in connection with [Theorem 1](#)), [7–9](#) (in connection with [Theorem 2](#)), and [10–12](#) (in connection with [Theorem 3](#)). The cases we have mentioned in this work are precisely those corresponding to $\theta = 1$. $\theta = 1/2$ gives rise to *midpoint rule*

approximations. Since the statements of the theorems for arbitrary θ are more involved, we do not discuss them here and refer the reader to the original works.

2. An important and interesting point to note concerning the asymptotic expansions given in all these sources is that, *provided $f(x)$ is infinitely differentiable in the open interval (a, b) [i.e., $f \in C^\infty(a, b)$], they are all determined completely by the asymptotic expansions of $f(x)$ as $x \rightarrow a +$ and $x \rightarrow b -$; nothing else is needed. This conclusion is valid for all our developments in the next sections too.*



3. ERROR EXPANSIONS FOR GAUSS–LEGENDRE QUADRATURE

In the preceding section we were concerned with the trapezoidal rule approximations or their modifications to finite-range integrals with or without endpoint singularities of algebraic–logarithmic types. We now turn to their approximation by the Gauss–Legendre quadrature. For convenience of notation, we will consider integrals on the (standard) interval $(-1, 1)$. We also assume that these integrals exist in the regular sense. Thus the integral $I[f]$ of $f(x)$ and the corresponding n -point Gauss–Legendre quadrature formula are

$$I[f] = \int_{-1}^1 f(x) dx \quad \text{and} \quad G_n[f] = \sum_{i=1}^n w_{ni} f(x_{ni}), \quad (27)$$

where x_{ni} are the abscissas [i.e., the zeros of $P_n(x)$, the n -th Legendre polynomial] and w_{ni} are the corresponding weights. Recall that the x_{ni} are all in the open interval $(-1, 1)$. This guarantees that $G_n[f]$ is well defined when $f(x)$ is continuous on $(-1, 1)$ no matter how $f(x)$ behaves at $x = \pm 1$. The functions $f(x)$ we consider have this property.

It is known that if $f \in C^\infty[-1, 1]$, then the error $G_n[f] - I[f]$ tends to zero as $n \rightarrow \infty$ faster than all negative powers of n , that is, $G_n[f] - I[f] = o(n^{-\mu})$ as $n \rightarrow \infty$ for every $\mu > 0$. In particular, when $f(z)$ is analytic in an open set of the z -plane that contains the interval $[-1, 1]$ in its interior, there holds $G_n[f] - I[f] = O(e^{-\sigma n})$ as $n \rightarrow \infty$ for some $\sigma > 0$; see Davis and Rabinowitz [Ref. 18, p. 312].

When $f(x)$ has integrable endpoint singularities, however, the error tends to zero slowly, its rate of decay depending on the strength of the singularities. For example, when $f(x) = (1-x)^\alpha g(x)$, with $\Re \alpha > -1$ but $\alpha \neq 0, 1, \dots$, and

$g \in C^\infty[-1, 1]$, it is known that the error is $O(n^{-2\alpha-2})$ as $n \rightarrow \infty$; see Davis and Rabinowitz [Ref. 18, p. 313]. A complete asymptotic expansion for $G_n[f]$ for this case was first derived by Verlinden in Ref. 19, theorem 1, by imposing on $f(z)$ some analyticity conditions. This expansion is reproduced in Theorem 4 that follows.

Theorem 4. *Let $f(x) = (1-x)^\alpha g(x)$, with $\Re\alpha > -1$ but $\alpha \neq 0, 1, \dots$, and $g(z)$ analytic in an open set that contains the interval $[-1, 1]$ in its interior. Then $G_n[f]$ has the asymptotic expansion*

$$G_n[f] \sim I[f] + \sum_{k=1}^{\infty} a_k h^{\alpha+k} \quad (n \rightarrow \infty); \quad h = (n+1/2)^{-2}. \quad (28)$$

Here, a_k are some constants independent of n .

Verlinden also considers $G_n[f]$ for a more general case involving an algebraic–logarithmic endpoint singularity, namely, $f(x) = \log(1-x)(1-x)^\alpha g(x)$, and shows that the asymptotic expansion of $G_n[f]$ in this case is obtained by differentiating that of Theorem 4 with respect to α term by term.

We now apply Verlinden’s theorem (Theorem 4) with $g(x) \equiv 1$.

Theorem 5. *Let $f_\omega^\pm(x) = (1 \pm x)^\omega$, $\Re\omega > -1$, but ω not an integer. Then $I[f_\omega^+] = I[f_\omega^-] = 2^{\omega+1}/(\omega+1)$ and $G_n[f_\omega^+] = G_n[f_\omega^-]$ and*

$$G_n[f_\omega^\pm] \sim I[f_\omega^\pm] + \sum_{k=1}^{\infty} u_k(\omega) h^{\omega+k} \quad (n \rightarrow \infty), \quad (29)$$

for some functions $u_k(\omega)$ that are analytic in ω for $\Re\omega > -1$.^c The $u_k(\omega)$ are the same for both $f_\omega^+(x)$ and $f_\omega^-(x)$. If ω is a nonnegative integer, then $G_n[f_\omega^\pm] = I[f_\omega^\pm]$ for $n \geq (\omega+1)/2$ and $u_k(\omega) = 0$ for all $k \geq 1$.^d

In Theorem 6 that we state next we consider functions $f(x)$ that have arbitrary algebraic–logarithmic endpoint singularities at one or both endpoints ± 1 that are basically as those in Theorem 3, without any analyticity assumption being made. Thus the class of functions we consider is much more general than that considered in Ref. 19 and contains the latter as a subclass. This theorem is due to Sidi,²⁰ and its proof makes use of Theorem 5.

^c So far, no simple expression for the functions $u_k(\omega)$ is known.

^d This follows from the fact that $G_n[f] = I[f]$ for all functions $f(x)$ that are polynomials of degree at most $2n-1$.

Theorem 6. Assume that $f \in C^\infty(-1, 1)$ and let $h = (n+1/2)^{-2}$ as before, and define $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. Let the functions $u_k(\omega)$ be precisely those that appear in Theorem 5. We consider the following two cases:

1. Let $f(x)$ have asymptotic expansions as $x \rightarrow -1$ (from the right) and as $x \rightarrow 1$ (from the left) as in

$$\begin{aligned} f(x) &\sim \sum_{s=0}^{\infty} c_s (1+x)^{\gamma_s} \quad (x \rightarrow -1+), \\ f(x) &\sim \sum_{s=0}^{\infty} d_s (1-x)^{\delta_s} \quad (x \rightarrow 1-), \end{aligned} \tag{30}$$

where γ_s and δ_s are in general complex and satisfy

$$\begin{aligned} -1 < \Re\gamma_0 \leq \Re\gamma_1 \leq \Re\gamma_2 \leq \dots; \quad \lim_{s \rightarrow \infty} \Re\gamma_s = +\infty, \\ -1 < \Re\delta_0 \leq \Re\delta_1 \leq \Re\delta_2 \leq \dots; \quad \lim_{s \rightarrow \infty} \Re\delta_s = +\infty. \end{aligned} \tag{31}$$

Assume also that the asymptotic expansions in (30) are differentiable infinitely many times. Then

$$\begin{aligned} G_n[f] \sim I[f] &+ \sum_{s=0}^{\infty} c_s \sum_{\substack{k=1 \\ \gamma_s \notin \mathbb{Z}^+}}^{\infty} u_k(\gamma_s) h^{\gamma_s+k} \\ &+ \sum_{s=0}^{\infty} d_s \sum_{\substack{k=1 \\ \delta_s \notin \mathbb{Z}^+}}^{\infty} u_k(\delta_s) h^{\delta_s+k} \quad (n \rightarrow \infty). \end{aligned} \tag{32}$$

2. More generally, let $f(x)$ have asymptotic expansions as $x \rightarrow -1$ (from the left) and as $x \rightarrow 1$ (from the right) as in

$$\begin{aligned} f(x) &\sim \sum_{s=0}^{\infty} P_s(\log(1+x))(1+x)^{\gamma_s} \quad (x \rightarrow -1+), \\ f(x) &\sim \sum_{s=0}^{\infty} Q_s(\log(1-x))(1-x)^{\delta_s} \quad (x \rightarrow 1-), \end{aligned} \tag{33}$$

where $P_s(y)$ and $Q_s(y)$ are arbitrary polynomials in y , and γ_s and δ_s are in general complex and satisfy

$$\begin{aligned} -1 < \Re\gamma_0 \leq \Re\gamma_1 \leq \Re\gamma_2 \leq \dots; \quad \lim_{s \rightarrow \infty} \Re\gamma_s = +\infty, \\ -1 < \Re\delta_0 \leq \Re\delta_1 \leq \Re\delta_2 \leq \dots; \quad \lim_{s \rightarrow \infty} \Re\delta_s = +\infty, \end{aligned} \tag{34}$$

such that

$$\begin{aligned} \Re\gamma_{s+1} = \Re\gamma_s &\Rightarrow \deg P_{s+1} \leq \deg P_s \text{ and} \\ \Re\delta_{s+1} = \Re\delta_s &\Rightarrow \deg Q_{s+1} \leq \deg Q_s. \end{aligned} \quad (35)$$

Assume also that the asymptotic expansions in (33) are differentiable infinitely many times. Then

$$\begin{aligned} G_n[f] \sim I[f] &+ \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} P_s(D_{\gamma_s}) [u_k(\gamma_s) h^{\gamma_s+k}] \\ &+ \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} Q_s(D_{\delta_s}) [u_k(\delta_s) h^{\delta_s+k}] \quad (n \rightarrow \infty). \end{aligned} \quad (36)$$

For the precise meaning of the terms $P_s(D_{\gamma_s}) [u_k(\gamma_s) h^{\gamma_s+k}]$ and $Q_s(D_{\delta_s}) [u_k(\delta_s) h^{\delta_s+k}]$, we refer the reader to Remark 7 following the statement of Theorem 3.

Note that, with $f(x)$ as in part 1 of Theorem 6, when $\gamma_0, \delta_0 \notin \mathbb{Z}^+$, we have at worst $G_n[f] - I[f] = O(n^{-2(\alpha+1)})$ as $n \rightarrow \infty$, where $\alpha = \min\{\Re\gamma_0, \Re\delta_0\}$.

When $f(x)$ is infinitely differentiable in the (closed) interval $[-1, 1]$, then $f(x)$ is precisely of the form described in (30) with the γ_s and δ_s in \mathbb{Z}^+ , which implies that $u_k(\gamma_s) = 0$ and $u_k(\delta_s) = 0$ for all $k = 1, 2, \dots$. That is, the asymptotic expansion in (32) is empty; therefore, $G_n[f] - I[f] = O(n^{-\mu})$ as $n \rightarrow \infty$ for every $\mu > 0$. This means that the error in $G_n[f]$ tends to zero faster than every negative power of n , which is a known result mentioned also above.

As is the case concerning the E–M expansion and all of its generalizations, an important and interesting point to note concerning the asymptotic expansions given in all these sources is that, provided $f(x)$ is infinitely differentiable in the open interval $(-1, 1)$ [i.e., $f \in C^\infty(-1, 1)$], they are all determined completely by the asymptotic expansions of $f(x)$ as $x \rightarrow -1 +$ and $x \rightarrow 1 -$; nothing else is needed.



4. ASYMPTOTICS OF LEGENDRE POLYNOMIALS AND LEGENDRE SERIES COEFFICIENTS

Orthogonal polynomial expansions of functions appear in most branches of science and engineering and their partial sums serve as approximations to the functions involved. Of course, the quality of these

approximations depends on the behavior of the coefficients in these expansions, and the behavior of the coefficients is determined by the smoothness properties of the functions being approximated. Here we are concerned with Legendre polynomial expansions on the interval $(-1, 1)$ specifically. As usual, $P_n(x)$ is the n -th Legendre polynomial standardized such that $P_n(1) = 1$, so that

$$\int_{-1}^1 P_m(x)P_n(x)dx = \frac{1}{n+1/2}\delta_{m,n}, \quad m, n = 0, 1, \dots \quad (37)$$

4.1 Asymptotics of Legendre Polynomials and Related Integrals

We begin with the asymptotic analysis of Legendre polynomials in the open interval $(-1, 1)$. It is known that, for large n and for $x \in (-1, 1)$, the Legendre polynomial $P_n(x)$ behaves like a trigonometric function, as in

$$P_n(\cos\theta) = \left(\frac{2}{\pi n \sin\theta}\right)^{1/2} \sin\left(n\theta + \frac{\theta}{2} + \frac{\pi}{4}\right) + O\left(n^{-3/2}\right) \quad (n \rightarrow \infty), \quad x = \cos\theta. \quad (38)$$

The following complete asymptotic expansion is given in Sidi [Ref. 21, theorem 3.2]:

Theorem 7. *There exist analytic functions $\phi_k(z)$ that are regular for $|z| = 1$, $z \neq \pm 1$, such that, with $\hat{n} = n + 1/2$ and arbitrary fixed $\epsilon \in (0, \pi/2)$,*

$$P_n(\cos\theta) \sim \Re \left\{ e^{i\hat{n}\theta} \sum_{k=0}^{\infty} \frac{\phi_k(e^{i\theta})}{\hat{n}^{k+1/2}} \right\} \quad (n \rightarrow \infty), \quad \text{uniformly for } \epsilon \leq \theta \leq \pi - \epsilon. \quad (39)$$

That is, for each $p = 0, 1, \dots$, and for $0 < \theta < \pi$, there holds

$$P_n(\cos\theta) = \Re \left\{ e^{i\hat{n}\theta} \left[\sum_{k=0}^{p-1} \frac{\phi_k(e^{i\theta})}{\hat{n}^{k+1/2}} + R_{p,n}(\theta) \right] \right\}, \quad (40)$$

where

$$R_{p,n}(\theta) = O\left(\hat{n}^{-p-1/2}\right) \quad (n \rightarrow \infty), \quad \text{uniformly for } \epsilon \leq \theta \leq \pi - \epsilon. \quad (41)$$

Actually, with $D_\theta = d/d\theta$, we have

$$\begin{aligned}
 \phi_k(e^{i\theta}) &= (-1)^k \frac{2}{\pi^{1/2}} \binom{-1/2}{k} e^{i(\theta-\pi)/2} \sum_{s=0}^k \binom{k}{s} B_{k-s}^{(1/2)} \cdot \left\{ \left(\frac{i}{2} D\theta\right)^s \left[(1 - e^{i2\theta})^{-1/2} \right] \right\} \\
 &= (-1)^k \frac{2}{\pi^{1/2}} \binom{-1/2}{k} e^{i(\theta-\pi)/2} B_k^{(1/2)} \left(\frac{i}{2} D\theta\right) \left[(1 - e^{i2\theta})^{-1/2} \right].
 \end{aligned}
 \tag{42}$$

Here $B_s^{(\sigma)}$ and $B_s^{(\sigma)}(u)$ are generalized Bernoulli numbers and polynomials.^e

It is easy to see that (38) is obtained from the first ($k = 0$) term of the summation in (40).

An expansion of the form similar to that in (40) is given in Szegő [Ref. 23, p. 196, theorem 8.21.9], and this expansion involves the powers $n^{-k-1/2}$ and does not provide the coefficients $\phi_k(z)$ explicitly. This should be compared with our expansion in (40) that involves the powers $(n+1/2)^{-k-1/2}$, for which the $\phi_k(z)$ are given explicitly.

The next theorem concerns the asymptotics of integrals of the form $\int_c^d f(x) P_n(x) dx$ with $-1 < c < d < 1$ and it is given in Ref. 21, theorem 4.2.

Theorem 8. Let $-1 < c < d < 1$, and assume that $f \in C^\infty(c, d)$. Let $\alpha = \cos^{-1} d$ and $\beta = \cos^{-1} c$; clearly $0 < \alpha < \beta < \pi$. Assume that, as $x \rightarrow c+$ and as $x \rightarrow d-$, $f(x)$ is such that $F(\theta) = \sin \theta \cdot f(\cos \theta)$ has the asymptotic expansions

$$\begin{aligned}
 F(\theta) &\sim \sum_{s=0}^{\infty} U_s(\theta - \alpha)^{\rho_s} \quad (\theta \rightarrow \alpha +); \quad \alpha = \cos^{-1} d > 0, \\
 F(\theta) &\sim \sum_{s=0}^{\infty} V_s(\beta - \theta)^{\sigma_s} \quad (\theta \rightarrow \beta -); \quad \beta = \cos^{-1} c < \pi,
 \end{aligned}
 \tag{43}$$

where

$$\begin{aligned}
 -1 < \Re \rho_0 \leq \Re \rho_1 \leq \Re \rho_2 \leq \dots; \quad \lim_{s \rightarrow \infty} \Re \rho_s = \infty, \\
 -1 < \Re \sigma_0 \leq \Re \sigma_1 \leq \Re \sigma_2 \leq \dots; \quad \lim_{s \rightarrow \infty} \Re \sigma_s = \infty,
 \end{aligned}
 \tag{44}$$

^e The generalized Bernoulli polynomials $B_s^{(\sigma)}(u)$ are defined via (see, for example, Andrews et al. [Ref. 22, p. 615])

$$\left(\frac{t}{e^t - 1}\right)^\sigma e^{ut} = \sum_{s=0}^{\infty} B_s^{(\sigma)}(u) \frac{t^s}{s!}, \quad |t| < 2\pi.$$

They satisfy $B_s^{(\sigma)}(\sigma - u) = (-1)^s B_s^{(\sigma)}(u)$; hence $B_s^{(\sigma)}(\sigma/2) = 0$ for $s = 1, 3, 5, \dots$. $B_s^{(\sigma)}(0)$ are called the generalized Bernoulli numbers and are denoted by $B_s^{(\sigma)}$. Note that $B_0^{(\sigma)} = 1$ for all σ . In addition, $B_k^{(\sigma)}(u) = \sum_{s=0}^k \binom{k}{s} B_{k-s}^{(\sigma)} u^s$ for all k .

and U_s and V_s are nonzero constants. Assume also that these asymptotic expansions can be differentiated termwise an infinite number of times. With the functions $\phi_k(z)$ as in [Theorem 7](#), for arbitrary $\theta \in [\alpha, \beta]$, let

$$\phi_{kj}(\theta) = \frac{1}{j!} \frac{d^j}{d\theta^j} \phi_k(e^{i\theta}), \quad j, k = 0, 1, \dots, \quad (45)$$

and

$$\begin{aligned} G_\mu^{(+)}(\theta; \omega) &= \frac{1}{2} \sum_{\substack{j, k \geq 0 \\ j+k=\mu}} i^{\omega+j+1} \phi_{kj}(\theta) \Gamma(\omega+j+1), \\ \hat{G}_\mu^{(+)}(\theta; \omega) &= \frac{1}{2} \sum_{\substack{j, k \geq 0 \\ j+k=\mu}} (-i)^{\omega+j+1} \overline{\phi_{kj}(\theta)} \Gamma(\omega+j+1), \\ G_\mu^{(-)}(\theta; \omega) &= \frac{1}{2} \sum_{\substack{j, k \geq 0 \\ j+k=\mu}} (-1)^j (-i)^{\omega+j+1} \phi_{kj}(\theta) \Gamma(\omega+j+1), \\ \hat{G}_\mu^{(-)}(\theta; \omega) &= \frac{1}{2} \sum_{\substack{j, k \geq 0 \\ j+k=\mu}} (-1)^j i^{\omega+j+1} \overline{\phi_{kj}(\theta)} \Gamma(\omega+j+1). \end{aligned} \quad (46)$$

Then, with $\hat{n} = n + 1/2$, as $n \rightarrow \infty$,

$$\begin{aligned} \int_c^d f(x) P_n(x) dx &\sim e^{i\hat{n}\alpha} \sum_{s=0}^{\infty} U_s \sum_{\mu=0}^{\infty} \frac{G_\mu^{(+)}(\alpha; \rho_s)}{\hat{n}^{\rho_s + \mu + 3/2}} + e^{-i\hat{n}\alpha} \sum_{s=0}^{\infty} U_s \sum_{\mu=0}^{\infty} \frac{\hat{G}_\mu^{(+)}(\alpha; \rho_s)}{\hat{n}^{\rho_s + \mu + 3/2}} \\ &+ e^{i\hat{n}\beta} \sum_{s=0}^{\infty} V_s \sum_{\mu=0}^{\infty} \frac{G_\mu^{(-)}(\beta; \sigma_s)}{\hat{n}^{\sigma_s + \mu + 3/2}} + e^{-i\hat{n}\beta} \sum_{s=0}^{\infty} V_s \sum_{\mu=0}^{\infty} \frac{\hat{G}_\mu^{(-)}(\beta; \sigma_s)}{\hat{n}^{\sigma_s + \mu + 3/2}}. \end{aligned} \quad (47)$$

Note again that the asymptotic expansion in (47) is determined completely by the asymptotic expansions in (43), nothing else being needed. In addition, it is the sum of four asymptotic expansions that are multiplied by $\cos n\alpha$, $\sin n\alpha$, $\cos n\beta$, and $\sin n\beta$.

4.2 Asymptotics of Legendre Series Coefficients

Let $\sum_{n=0}^{\infty} e_n[f] P_n(x)$ be the Legendre series of the function $f(x)$ on $(-1, 1)$, where

$$e_n[f] = (n + 1/2) \int_{-1}^1 P_n(x) f(x) dx, \quad n = 0, 1, \dots \quad (48)$$

It is known that when $f(x)$ and $|f(x)|^2$ are integrable on $(-1, 1)$, we have (see, for example, Szegő²³ or Freud²⁴),

$$e_n[f] = o(\sqrt{n}) \quad (n \rightarrow \infty). \quad (49)$$

When $f \in C^r[-1, 1]$ for some integer $r \geq 0$, then (see Sidi²⁵, Introduction)

$$e_n[f] = O\left(n^{-r+1/2} \omega_{f^{(r)}}(2/n)\right) \quad (n \rightarrow \infty), \quad (50)$$

where $\omega_g(\delta)$ stands for the modulus of continuity of $g(x)$ on $[-1, 1]$. (For moduli of continuity, see, for example, Davis [Ref. 26, pp. 7–8], or Lorentz [Ref. 27, pp. 43–46].) Clearly, when $f(x)$ is continuously differentiable only r times on $[-1, 1]$, the best we can say about $e_n[f]$ is (50), and that the smaller r is, the slower the convergence of the series to $f(x)$ becomes. Of course, neither (49) nor (50) give us the best possible information about the behavior of $e_n[f]$ as $n \rightarrow \infty$.

From (50), it is easy to see that when $f \in C^\infty[-1, 1]$, $e_n[f]$ satisfies (51) for every $r > 0$, and this implies that $e_n[f]$ tends to zero as $n \rightarrow \infty$ faster than every negative power of n , that is,

$$e_n[f] = O(n^{-\mu}) \quad (n \rightarrow \infty), \quad \forall \mu > 0. \quad (51)$$

In particular, when $f(z)$ is analytic in an open set of the z -plane that contains the interval $[-1, 1]$ in its interior, there holds

$$e_n[f] = O(e^{-\sigma n}) \quad (n \rightarrow \infty), \quad \text{for some } \sigma > 0. \quad (52)$$

4.2.1 Asymptotics of $e_n[f]$ in Presence of Endpoint Singularities

We now turn to functions $f(x)$ that are infinitely differentiable in the (open) interval $(-1, 1)$ but can have regular or general singular behavior at one or both of the endpoints $x = \pm 1$ as described in Theorem 6. In Theorem 10, we present a complete asymptotic expansion as $n \rightarrow \infty$ for $e_n[f]$ that is derived in Sidi.²⁵

We begin with the following result that is analogous to Theorem 5. See Sidi [Ref. 25, theorem 2.1].

Theorem 9. *Let $f_\omega^\pm(x) = (1 \pm x)^\omega$, $\Re\omega > -1$, but ω not an integer. Then*

$$(-1)^n e_n[f_\omega^+] = e_n[f_\omega^-] = 2^\omega (2n + 1) (-\omega)_n / (\omega + 1)_{n+1}, \quad n = 0, 1, \dots, \quad (53)$$

and

$$(-1)^n e_n[f_\omega^+] = e_n[f_\omega^-] \sim \sum_{k=1}^{\infty} v_k(\omega) h^{\omega+k+1/2} \quad (n \rightarrow \infty); \quad h = (n+1/2)^{-2}, \quad (54)$$

for some functions $v_k(\omega)$ that are analytic in ω for $\Re \omega > -1$ that are given as in

$$v_k(\omega) = 2^{\omega+1} \frac{\Gamma(1+\omega)}{\Gamma(-\omega)} \frac{B_{2k}^{(\sigma)}(\sigma/2)}{(2k)!} \frac{\Gamma(2k+2\omega+2)}{\Gamma(2\omega+2)}, \quad (55)$$

$$k = 0, 1, \dots, \quad \sigma = -2\omega - 1.$$

Here $B_s^{(\sigma)}(u)$ is the s -th generalized Bernoulli polynomial. When $\omega \in \mathbb{Z}^+$, there holds $v_k(\omega) = 0$ for each $k = 0, 1, \dots$; in this case, we also have $e_n[f_\omega^\pm] = 0$ for all $n > \omega$.

In [Theorem 10](#), we use the notation of [Theorem 6](#).

Theorem 10. Assume that $f \in C^\infty(-1, 1)$ and $h = (n+1/2)^{-2}$, and define $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. Let the functions $v_k(\omega)$ be precisely those that appear in [Theorem 9](#). We consider the following two cases:

1. Let $f(x)$ have asymptotic expansions as $x \rightarrow -1$ and as $x \rightarrow 1$ precisely as in (30) and (31) of [Theorem 6](#), and assume that these asymptotic expansions are infinitely differentiable. Then

$$e_n[f] \sim \sum_{s=0}^{\infty} c_s \sum_{\substack{k=1 \\ \gamma_s \notin \mathbb{Z}^+}}^{\infty} v_k(\gamma_s) h^{\gamma_s+k+1/2} \\ + (-1)^n \sum_{s=0}^{\infty} d_s \sum_{\substack{k=1 \\ \delta_s \notin \mathbb{Z}^+}}^{\infty} v_k(\delta_s) h^{\delta_s+k+1/2} \quad (n \rightarrow \infty). \quad (56)$$

2. More generally, let $f(x)$ have asymptotic expansions as $x \rightarrow -1$ and as $x \rightarrow 1$ precisely as in (33)–(35) of [Theorem 6](#), and assume that these asymptotic expansions are infinitely differentiable. Then

$$e_n[f] \sim \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} P_s(D_{\gamma_s}) [v_k(\gamma_s) h^{\gamma_s+k+1/2}] \\ + (-1)^n \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} Q_s(D_{\delta_s}) [v_k(\delta_s) h^{\delta_s+k+1/2}] \quad (n \rightarrow \infty). \quad (57)$$

For the precise meaning of the terms $P_s(D_{\gamma_s})[v_k(\gamma_s)h^{\gamma_s+k}]$ and $Q_s(D_{\delta_s})[v_k(\delta_s)h^{\delta_s+k}]$, we refer the reader to [Remark 7](#) following the statement of [Theorem 3](#).

Note that, with $f(x)$ as in part 1 of [Theorem 10](#), when $\gamma_0, \delta_0 \notin \mathbb{Z}^+$, we have at worst $e_n[f] = O(n^{-2(\alpha+1)})$ as $n \rightarrow \infty$, where $\alpha = \min\{\Re\gamma_0, \Re\delta_0\}$.

4.2.2 Asymptotics of $e_n[f]$ in Presence of Interior and Endpoint Singularities

[Theorem 10](#) can be extended to situations in which $f(x)$ has a number of singularities in the (open) interval $(-1, 1)$. This problem has been treated in detail in Sidi²¹ by assuming algebraic singularities. Since this treatment is complicated, we will be content with a brief description of the assumptions and a sketch of the results.

Let us assume first that $f(x)$ has asymptotic expansions at $x = \pm 1$ as given in (30). Let us assume also that $f(x)$ has algebraic singularities in $(-1, 1)$ at the points $x_1 > x_2 > \dots > x_m$ and that $f \in C^\infty(-1, 1)$ everywhere else in $(-1, 1)$. Let us also assume that $f(x)$ has asymptotic expansions as $x \rightarrow x_r \pm$, that is, as $x \rightarrow x_r$ from the right and from the left, of the forms

$$f(x) \sim \sum_{s=0}^{\infty} W_{rs}^{(\pm)} |x - x_r|^{\delta_{rs}^{(\pm)}} \quad (x \rightarrow x_r \pm), \quad (58)$$

and let us allow these expansions to be different. Making the change of variable $x = \cos \theta$ in (48), we can express $e_n[f]$ in the form

$$e_n[f] = (n+1/2) \int_0^\pi F(\theta) P_n(\cos \theta) d\theta; \quad F(\theta) = \sin \theta \cdot f(\cos \theta). \quad (59)$$

Then $F \in C^\infty(0, \pi)$ except at $\theta_1 < \theta_2 < \dots < \theta_m$, where $\theta_r = \cos^{-1} x_r$, $r = 1, \dots, m$, and it has asymptotic expansions as $\theta \rightarrow \theta_r \pm$, of the form

$$F(\theta) \sim \sum_{s=0}^{\infty} T_{rs}^{(\pm)} |\theta - \theta_r|^{\gamma_{rs}^{(\pm)}} \quad (\theta \rightarrow \theta_r \pm), \quad (60)$$

where the $\gamma_{rs}^{(\pm)}$ satisfy

$$-1 < \Re\gamma_{r0}^{(\pm)} \leq \Re\gamma_{r1}^{(\pm)} \leq \Re\gamma_{r2}^{(\pm)} \dots; \quad \lim_{s \rightarrow \infty} \Re\gamma_{rs}^{(\pm)} = 0. \quad (61)$$

Note that $\gamma_{rs}^{(\pm)}$ are determined by the $\delta_{rs}^{(\pm)}$. The asymptotic expansion of $e_n[f]$ can be obtained by using [Theorems 8](#) and [10](#) and so-called neutralizing

functions. We do not go into any detail here and refer the reader to Sidi [Ref. 21, theorem 4.2]. We only mention that each singular point $\theta = \theta_r$ contributes two asymptotic expansions in powers of $\hat{n} = n + 1/2$, one multiplied by $e^{i\hat{n}\theta_r}$ and the other multiplied by $e^{-i\hat{n}\theta_r}$. The contribution from the endpoints $x = \pm 1$ is precisely that given in Theorem 9. See Ref. 21 for details.

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