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ORIGINAL PAPER

Acceleration of convergence of some infinite sequences $\{A_n\}$ whose asymptotic expansions involve fractional powers of n via the $\tilde{d}^{(m)}$ transformation

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Abstract

In this paper, we discuss the application of the author's $\tilde{d}^{(m)}$ transformation to accelerate the convergence of infinite series $\sum_{n=1}^{\infty} a_n$ when the terms a_n have asymptotic expansions that can be expressed in the form

$$a_n \sim (n!)^{s/m} \exp\left[\sum_{i=0}^{m} q_i n^{i/m}\right] \sum_{i=0}^{\infty} w_i n^{\gamma-i/m} \quad \text{as } n \to \infty, \quad s \text{ integer.}$$

We discuss the implementation of the $\tilde{d}^{(m)}$ transformation via the recursive W-algorithm of the author. We show how to apply this transformation and how to assess in a reliable way the accuracies of the approximations it produces, whether the series converge or they diverge. We classify the different cases that exhibit unique numerical stability issues in floating-point arithmetic. We show that the $\tilde{d}^{(m)}$ transformation can also be used efficiently to accelerate the convergence of infinite products $\prod_{n=1}^{\infty}(1+v_n)$, where $v_n \sim \sum_{i=0}^{\infty} e_i n^{-t/m-i/m}$ as $n \to \infty$, $t \geq m+1$ an integer. Finally, we give several numerical examples that attest the high efficiency of the $\tilde{d}^{(m)}$ transformation for the different cases.

Keywords Acceleration of convergence · Extrapolation · Infinite series · Infinite products · Asymptotic expansions · Fractional powers · $\tilde{d}^{(m)}$ transformation · W-algorithm

Mathematics subject classification (2010) 40A05 · 40A25 · 40A20 · 65B05 · 65B10

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1 Introduction

The summation of infinite series $\sum_{n=1}^{\infty} a_n$, where the terms a_n are in general complex and have asymptotic expansions (as $n \to \infty$) involving powers of $n^{-1/m}$ for positive integers m, has been of some interest. Due to their complex analytical nature, however, the rigorous study of such series has been the subject of very few works. See Birkhoff [2] and Birkhoff and Trjitzinsky [3]. For a brief summary of these works, see Wimp [23, 24, Section 1.7].

In this work, we deal with those infinite series $\sum_{n=1}^{\infty} a_n$, whether convergent or divergent, for which $\{a_n\}$ belong to a class of sequences denoted $\tilde{\mathbf{b}}^{(m)}$. These series were first studied in detail in Sidi [18, Section 6.6], where an extrapolation method denoted the $\tilde{d}^{(m)}$ transformation to accelerate their convergence (actually, to accelerate the convergence of the sequence $\{A_n\}$ of the partial sums $A_n = \sum_{k=1}^{n} a_k$, $n = 1, 2, \ldots$) was also developed. This transformation is very effective also when these series diverge; in such cases, it produces approximations to the antilimits of the series treated. Practically speaking, a sequence $\{a_n\}$ is in $\tilde{\mathbf{b}}^{(m)}$, $m \geq 1$ being an integer, if a_n has an asymptotic expansion that *can be expressed* in the form

$$a_n \sim [\Gamma(n)]^{s/m} \exp[Q(n)] \sum_{i=0}^{\infty} w_i n^{\gamma - i/m} \quad \text{as } n \to \infty, \tag{1.1}$$

where

1. $\Gamma(z)$ is the gamma function.
2. s is an arbitrary integer, positive, negative, or zero.
3. $Q(n)$ is either identically zero or is a polynomial of degree at most m in $n^{1/m}$, expressed as in

$$Q(n) = \sum_{i=0}^{m-1} \theta_i n^{1-i/m}, \tag{1.2}$$

   $\theta_0, \theta_1, \ldots, \theta_{m-1}$ being real or complex constants.[1]
4. γ is an arbitrary real or complex number.

In the special case of $m = 1$, either $Q(n) = \theta_0 n$ with $\theta_0 \neq 0$ or $Q(n) \equiv 0$, and (1.1) assumes the form

$$a_n \sim [\Gamma(n)]^{s} \zeta^n \sum_{i=0}^{\infty} w_i n^{\gamma - i} \quad \text{as } n \to \infty, \tag{1.3}$$

with (i) $\zeta = 1$ if $Q(n) \equiv 0$ and (ii) $\zeta = e^{\theta_0} \neq 1$ if $Q(n) = \theta_0 n$ with $\theta_0 \neq 0$. Here, we note that the class $\tilde{\mathbf{b}}^{(1)}$ is simply the class denoted $\mathbf{b}^{(1)}$, which is a special case and the simplest prototype of the collection of sequence classes $\mathbf{b}^{(m)}$, $m = 1, 2, \ldots,$ originally introduced in Levin and Sidi [10] and studied extensively in Sidi

[1] Clearly, $Q(n) \equiv 0$ takes place when $\theta_i = 0$, $i = 0, 1, \ldots, m-1$.

[18, Chapter 6].[2] In this connection, we mention that the t, u, and v transformations of Levin [9] and the $d^{(1)}$ transformation of Levin and Sidi [10] are very effective convergence acceleration methods for infinite series $\sum_{n=1}^{\infty} a_n$ with $\{a_n\} \in \mathbf{b}^{(1)}$.

In this work, we shall deal with the class $\tilde{\mathbf{b}}^{(m)}$, $m \geq 1$ being arbitrary. We shall use the notation of [18, Section 6.6] throughout. Comparing (1.1)–(1.2) with (1.3), and judging also from Theorem 2.5, we realize that sequences in $\tilde{\mathbf{b}}^{(m)}$ with $m \geq 2$ have a richer and more interesting mathematical structure than those in $\tilde{\mathbf{b}}^{(1)} = \mathbf{b}^{(1)}$. As will also be clear from the numerical examples in Section 5, depending on whether a_n in (1.1) is such that

(i) $s = 0$ and $Q(n) \equiv 0$ and $\gamma \neq -1 + i/m$, $i = 0, 1, \ldots,$ or
(ii) $s = 0$ and $Q(n) \not\equiv 0$, with $\theta_0 \neq 0$ and γ is arbitrary, or
(iii) $s = 0$ and $Q(n) \not\equiv 0$, with $\theta_0 = \cdots = \theta_{r-1} = 0$ and $\theta_r \neq 0$ for some $r \in \{1, \ldots, m-1\}$, and γ is arbitrary, or
(iv) $s \neq 0$ ($s < 0$ or $s > 0$) and $Q(n)$ is arbitrary [$Q(n) \equiv 0$ or $Q(n) \not\equiv 0$], and γ is arbitrary, or
(v) a_n is as in any one of the cases (i)–(iv) (with real θ_0), multiplied by $(-1)^n$,

the series $\sum_{n=1}^{\infty} a_n$ exhibit different convergence and numerical stability properties when convergence acceleration methods are applied to them in finite-precision (floating-point) arithmetic. In addition, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

The contents of this paper are arranged as follows: In the next section, we summarize the asymptotic properties of sequences $\{a_n\}$ in $\tilde{\mathbf{b}}^{(m)}$ for arbitrary m. In Section 3, (i) we recall the $\tilde{d}^{(m)}$ transformation, (ii) we recall the issue of assessing the numerical stability of the approximations generated by it, (iii) we recall the W-algorithm of Sidi [14] as it is used for implementing the $\tilde{d}^{(m)}$ transformation, and (iv) we discuss how the W-algorithm can be extended for assessing in a very simple way the numerical stability of the approximations generated by the $\tilde{d}^{(m)}$ transformation simultaneously with their computation in finite-precision arithmetic. In Section 4, we illustrate Theorem 2.6, which concerns the asymptotic behavior of the partial sums $A_n = \sum_{k=1}^{n} a_k$ as $n \to \infty$, on the basis of which the $\tilde{d}^{(m)}$ transformation is developed, with some instructive examples. In Section 5, we illustrate with numerical examples of varying nature the remarkable effectiveness of the $\tilde{d}^{(m)}$ transformation on the series $\sum_{n=1}^{\infty} a_n$, where $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$, whether these converge or diverge. We also show how the $\tilde{d}^{(m)}$ transformation can be tuned for best numerical results. In Section 6, we consider the use of the $\tilde{d}^{(m)}$ transformation for computing some infinite products $\prod_{n=1}^{\infty}(1 + v_n)$, where $\{v_n\} \in \tilde{\mathbf{A}}_0^{(-t/m,m)}$, that is

$$v_n \sim \sum_{i=0}^{\infty} w_i n^{-t/m - i/m}, \quad t \geq m+1 \ \text{an integer}. \tag{1.4}$$

[2] The convergence of infinite series $\sum_{n=1}^{\infty} a_n$ with $\{a_n\} \in \mathbf{b}^{(m)}$, m being arbitrary, can be accelerated efficiently by using the $d^{(m)}$ transformation of Levin and Sidi [10], which can be implemented very economically via the recursive $\mathrm{W}^{(m)}$-algorithm of Ford and Sidi [8]. All this is studied in detail also in Sidi [18, Chapters 6 and 7].

We study the asymptotic behavior of the partial products $A_n = \prod_{k=1}^{n}(1 + v_k)$ as $n \to \infty$ and conclude that the $\tilde{d}^{(m)}$ transformation can be applied very efficiently to accelerate the convergence of the sequence of the partial products. In Section 7, we give numerical examples that illustrate the efficiency of the $\tilde{d}^{(m)}$ transformation on such infinite products.

Presently, there is no numerical experience with the issue of convergence acceleration of the infinite series described above in their most general form, that is, with arbitrary m, s, γ, and $Q(n)$. So far, the acceleration of the convergence of only a subset of such series, for which $s = 0$ and $Q(n) \equiv 0$ and $\sum_{n=1}^{\infty} a_n$ is convergent, has been considered in the literature; thus,

$$a_n \sim \sum_{i=0}^{\infty} w_i n^{\gamma - i/m} \quad \text{as } n \to \infty \quad \text{and} \quad \Re\gamma < -1, \tag{1.5}$$

in this subset: Sablonnière [12] has studied the application of (i) the iterated modified Δ^2-process and (ii) the iterated θ_2-algorithm of Brezinski [4], to the cases in which $m = 1, 2$ only. Van Tuyl [21, 22] has studied the application of (i) the iterated modified Δ^2-process, (ii) the iterated transformation of Lubkin [11], (iii) the θ-algorithm of Brezinski [4], (iv) a generalization of the ρ-algorithm of Wynn [25], (v) the u and v transformations of Levin [9], (vi) a generalization of the Neville table, and (vii) the $d^{(m)}$ transformation of Levin and Sidi [10]. The numerical results of [21] show that, with the exception of the u and v transformations, which are effective only when $m = 1$, the rest of the transformations are effective accelerators for all $m \geq 1$. (Note that the iterated θ_2-algorithm and iterated Lubkin transformation are identical.)

The modified Δ^2-process is due to Drummond [7] (see also Brezinski and Redivo-Zaglia [5]), while the generalized ρ-algorithm and the generalized Neville table are given in Van Tuyl [22]. For the Δ^2-process, which is due to Aitken [1], see Stoer and Bulirsch [20, Chapter 5] and Sidi [18, Chapter 15], for example. For discussions of the methods mentioned above, see also [18, Chapters 6, 15, 19, 20].

We note that to apply the modified Δ^2-process, the generalized ρ-algorithm, and the generalized Neville table, we need to know γ in (1.5). This is not the case when applying the iterated transformation of Lubkin, the θ-algorithm, the $d^{(m)}$ transformation, and the $\tilde{d}^{(m)}$ transformation.

Before proceeding further, we would like to emphasize that the $\tilde{d}^{(m)}$ transformation can be formulated such that it will be applicable without any modification and with success to *all* infinite series $\sum_{n=1}^{\infty} a_n$ where $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$, with *arbitrary* s, $Q(n)$, and γ, which do *not* have to be known. This is a very important feature of the $\tilde{d}^{(m)}$ transformation and of this work.

Finally, we mention that the works [12] and [22] deal only with the convergence issue of the transformations discussed in them; they do not consider the important issue of numerical stability when using floating-point (finite-precision) arithmetic.[3]

[3]Note that most of the methods mentioned above suffer from lack of numerical stability when applied to infinite series $\sum_{n=1}^{\infty} a_n$ with a_n behaving as in (1.5). In addition, there is no reliable way to assess the floating-point accuracies of the approximations they produce.

In our treatment of the $\tilde{d}^{(m)}$ transformation in Section 3 of this work, we emphasize this issue as follows: (i) we devise reliable zero-cost procedures for monitoring the numerical stability and predicting the maximum accuracy of the approximations produced at the time these are being computed and (ii) we overcome numerical instabilities by applying the $\tilde{d}^{(m)}$ transformation to properly sampled subsequences of the sequences $\{A_n\}$ of partial sums $A_n = \sum_{k=1}^{n} a_k$ via *arithmetic progression sampling (APS)* or *geometric progression sampling (GPS)*—two automatic sampling procedures originally proposed in Ford and Sidi [8]—that have been shown to be very effective. These are two additional important features of this work that differentiate it from all previous works.

2 Preliminaries

2.1 The function class $\tilde{\mathbf{A}}_0^{(\gamma,m)}$

We begin with the following definition:

Definition 2.1 ([18], **Definition 6.6.1)** A function $\alpha(x)$ defined for all large x is in the set $\tilde{\mathbf{A}}_0^{(\gamma,m)}$, m being a positive integer, if it has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim \sum_{i=0}^{\infty} \alpha_i x^{\gamma - i/m} \quad \text{as } x \to \infty. \tag{2.1}$$

In addition, if $\alpha_0 \neq 0$ in (2.1), then $\alpha(x)$ is said to belong to $\tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly. Here, γ is complex in general.[4]

Before going on, we state some properties of the sets $\tilde{\mathbf{A}}_0^{(\gamma,m)}$, whose verification we leave to the reader. We make repeated use of these properties in Sections 4 and 6.

1. $\tilde{\mathbf{A}}_0^{(\gamma,m)} \supset \tilde{\mathbf{A}}_0^{(\gamma-1/m,m)} \supset \tilde{\mathbf{A}}_0^{(\gamma-2/m,m)} \supset \cdots$, so that if $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$, then, for any positive integer k, $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma+k/m,m)}$ but not strictly. Conversely, if $\alpha \in \tilde{\mathbf{A}}_0^{(\delta,m)}$ but not strictly, then $\alpha \in \tilde{\mathbf{A}}_0^{(\delta-k/m,m)}$ strictly for a unique positive integer k.
2. If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly, then $\alpha \notin \tilde{\mathbf{A}}_0^{(\gamma-1/m,m)}$.
3. If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly, and $\beta(x) = \alpha(cx+d)$ for some arbitrary constants $c > 0$ and d, then $\beta \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly as well.
4. If $\alpha, \beta \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$, then $\alpha \pm \beta \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ as well. (This implies that the zero function is included in $\tilde{\mathbf{A}}_0^{(\gamma,m)}$.) If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ and $\beta \in \tilde{\mathbf{A}}_0^{(\gamma+k/m,m)}$ strictly for some positive integer k, then $\alpha \pm \beta \in \tilde{\mathbf{A}}_0^{(\gamma+k/m,m)}$ strictly.
5. If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ and $\beta \in \tilde{\mathbf{A}}_0^{(\delta,m)}$, then $\alpha\beta \in \tilde{\mathbf{A}}_0^{(\gamma+\delta,m)}$; if, in addition, $\beta \in \tilde{\mathbf{A}}_0^{(\delta,m)}$ strictly, then $\alpha/\beta \in \tilde{\mathbf{A}}_0^{(\gamma-\delta,m)}$.

[4]Clearly, if $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$, then $\alpha(x) = x^{\gamma}\beta(x)$, where $\beta \in \tilde{\mathbf{A}}_0^{(0,m)}$.

6. If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly, such that $\alpha(x) > 0$ for all large x, and we define $\theta(x) = [\alpha(x)]^{\xi}$, then $\theta \in \tilde{\mathbf{A}}_0^{(\gamma\xi,m)}$ strictly.
7. If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly and $\beta \in \tilde{\mathbf{A}}_0^{(k,1)}$ strictly for some positive integer k, such that $\beta(x) > 0$ for all large $x > 0$, and we define $\theta(x) = \alpha(\beta(x))$, then $\theta \in \tilde{\mathbf{A}}_0^{(k\gamma,m)}$ strictly.
8. If $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ (strictly), and $\beta(x) = \alpha(x+d) - \alpha(x)$ for an arbitrary constant $d \neq 0$, then $\beta \in \tilde{\mathbf{A}}_0^{(\gamma-1,m)}$ (strictly) when $\gamma \neq 0$. If $\alpha \in \tilde{\mathbf{A}}_0^{(0,m)}$, then $\beta \in \tilde{\mathbf{A}}_0^{(-1-1/m,m)}$.

Note that if $a_n = \alpha(n)$, $n = 1, 2, \ldots,$ where $\alpha \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$, then a_n is as in (1.5). Such sequences $\{a_n\}$ are therefore in the class $\tilde{\mathbf{b}}^{(m)}$.

The following theorem summarizes the summation properties of functions in $\tilde{\mathbf{A}}_0^{(\gamma,m)}$. It is also useful in proving Theorem 2.4.

Theorem 2.2 ([18], **Theorem 6.6.2)** *Let $g \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ strictly for some γ with $g(x) \sim \sum_{i=0}^{\infty} g_i x^{\gamma-i/m}$ as $x \to \infty$, and define $G(n) = \sum_{r=1}^{n-1} g(r)$. Then*

$$G(n) = b + c \log n + \tilde{G}(n), \tag{2.2}$$

where b and c are constants and $\tilde{G} \in \tilde{\mathbf{A}}_0^{(\gamma+1,m)}$.

1. *If $\gamma \neq -1$, then $\tilde{G} \in \tilde{\mathbf{A}}_0^{(\gamma+1,m)}$ strictly, while $\tilde{G} \in \tilde{\mathbf{A}}_0^{(-1/m,m)}$ if $\gamma = -1$.*
2. *If $\gamma + 1 \neq i/m,\ i = 0, 1, \ldots,$ then $c = 0$, and either (i) b is the limit of $G(n)$ as $n \to \infty$ if $\Re\gamma + 1 < 0$, or (ii) b is the antilimit of $G(n)$ as $n \to \infty$ if $\Re\gamma + 1 \geq 0$.*
3. *If $\gamma + 1 = k/m$ for some integer $k \geq 0$, then $c = g_k$.*

Finally,

$$\tilde{G}(n) = \sum_{\substack{i=0 \\ \gamma-i/m\neq-1}}^{m-1} \frac{g_i}{\gamma - i/m + 1} n^{\gamma-i/m+1} + O(n^{\gamma}) \ \text{ as } n \to \infty. \tag{2.3}$$

Before ending this section, we also note that the sets $\tilde{\mathbf{A}}_0^{(\gamma,m)}$ are most important building blocks of sequences $\{a_n\}$ in the class $\tilde{\mathbf{b}}^{(m)}$, to which we turn next.

2.2 The sequence class $\tilde{\mathbf{b}}^{(m)}$

With the classes $\tilde{\mathbf{A}}_0^{(\gamma,m)}$ already defined, we now go on to define the sequence class $\tilde{\mathbf{b}}^{(m)}$.

Definition 2.3 ([18], **Definition 6.6.3)** A sequence $\{a_n\}$ belongs to the class $\tilde{\mathbf{b}}^{(m)}$ if it satisfies a linear homogeneous difference equation of first order of the form $a_n = p(n)\Delta a_n$ with $p \in \tilde{\mathbf{A}}_0^{(q/m,m)}$ for some integer $q \leq m$. Here, $\Delta a_n = a_{n+1} - a_n$, $n = 1, 2, \ldots.$

We begin with the following general result:

Theorem 2.4 ([18], **Theorem 6.6.4**) (i) *Let* $a_{n+1} = c(n)a_n$ *such that* $c \in \tilde{\mathbf{A}}_0^{(\mu,m)}$ *strictly with* μ *in general complex. Then* a_n *is of the form*

$$a_n = [\Gamma(n)]^{\mu} \exp[Q(n)]\, n^{\gamma}\, w(n), \tag{2.4}$$

where $\Gamma(n)$ *is the gamma function and* $Q(n)$ *is a polynomial of degree at most* m *in* $n^{-1/m}$ *which we choose to write in the form*

$$Q(n) = \sum_{i=0}^{m-1} \theta_i n^{1-i/m}, \tag{2.5}$$

and

$$w \in \tilde{\mathbf{A}}_0^{(0,m)} \text{ strictly.} \tag{2.6}$$

Given that $c(n) \sim \sum_{i=0}^{\infty} c_i n^{\mu-i/m}$ *as* $n \to \infty$, *with* $c_0 \neq 0$, *we have*

$$e^{\theta_0} = c_0; \quad \theta_i = \frac{\epsilon_i}{1 - i/m}, \quad i = 1, \ldots, m-1; \; \gamma = \epsilon_m, \tag{2.7}$$

where the ϵ_i *are determined by* $c_0, c_1, \ldots, c_m$ *via*

$$\sum_{s=1}^{m} \frac{(-1)^{s+1}}{s} \left(\sum_{i=1}^{m} \frac{c_i}{c_0} z^i \right)^s = \sum_{i=1}^{m} \epsilon_i z^i + O(z^{m+1}) \text{ as } z \to 0. \tag{2.8}$$

(Note that $\theta_0 = 0$ *when* $c_0 = 1$.*)*

(ii) *The converse is also true, that is, if* a_n *is as in* (2.4)–(2.6), *then* $a_{n+1} = c(n)a_n$ *with* $c \in \tilde{\mathbf{A}}_0^{(\mu,m)}$ *strictly.*

(iii) *Finally, (a)* $\theta_1 = \cdots = \theta_{m-1} = 0$ *if and only if* $c_1 = \cdots = c_{m-1} = 0$, *and (b)* $\theta_1 = \cdots = \theta_{r-1} = 0$ *and* $\theta_r \neq 0$ *if and only if* $c_1 = \cdots = c_{r-1} = 0$ *and* $c_r \neq 0$, $r \in \{1, \ldots, m-1\}$.

Remark 1 Note that we can express (2.4) also in the form

$$a_n = [\Gamma(n)]^{\mu} \exp\left[\hat{Q}(n)\right] n^{\gamma}\, w(n)\, \zeta^n,$$

where

$$\hat{Q}(n) = \sum_{i=1}^{m-1} \theta_i n^{1-i/m}, \quad \zeta = c_0 = e^{\theta_0}.$$

Of course, $\zeta = 1$ when $c_0 = 1$ and $\zeta \neq 1$ when $c_0 \neq 1$.

The next theorem gives necessary and sufficient conditions for a sequence $\{a_n\}$ to be in $\tilde{\mathbf{b}}^{(m)}$. In this sense, it is a characterization theorem for sequences in $\tilde{\mathbf{b}}^{(m)}$. Theorem 2.4 becomes useful in the proof.

Theorem 2.5 ([18], **Theorem 6.6.5**) *A sequence* $\{a_n\}$ *is in* $\tilde{\mathbf{b}}^{(m)}$ *if and only if its members satisfy* $a_{n+1} = c(n)a_n$ *with* $c \in \tilde{\mathbf{A}}_0^{(s/m,m)}$ *for some integer* s *and* $c(n) \neq$

$1+O(n^{-1-1/m})$ as $n\to\infty$.[5] *Therefore, a_n is as in (2.4)–(2.6) with $\mu=s/m$, and this implies that $a_n=p(n)\Delta a_n$ with $p\in\tilde{\mathbf{A}}_0^{(\sigma,m)}$ strictly, where $\sigma=q/m$ and q is an integer $\leq m$. With $c(n)\sim\sum_{i=0}^{\infty}c_i n^{s/m-i/m}$ as $n\to\infty$, $c_0\neq 0$, we have the following specific cases:*

1. *When $s=0$, $c_0=1$, $c_1=\cdots=c_{m-1}=0$, and $c_m\neq 0$, which holds necessarily, we have $\sigma=1$ or $q=m$.*
   In this case, $a_n=n^{\gamma}w(n)$ with $\gamma=c_m\neq 0$. Hence $a_n=h(n)$, $h\in\tilde{\mathbf{A}}_0^{(\gamma,m)}$, $\gamma\neq 0$.
2. *When $s=0$, $c_0=1$, $c_1=\cdots=c_{r-1}=0$, and $c_r\neq 0$, $r<m$, we have $\sigma=r/m$ or $q=r$.*
   In this case, $a_n=\exp[Q(n)]n^{\gamma}w(n)$, $Q(n)=\sum_{i=r}^{m-1}\theta_i n^{1-i/m}$.
3. *When $s=0$, $c_0\neq 1$, we have $\sigma=0$ or $q=0$.*
   In this case, $a_n=\exp[Q(n)]n^{\gamma}w(n)$, $Q(n)=\sum_{i=0}^{m-1}\theta_i n^{1-i/m}$, $\theta_0\neq 0$.
4. *When $s<0$, we have $\sigma=0$ or $q=0$.*
   In this case, $a_n=[\Gamma(n)]^{s/m}\exp[Q(n)]n^{\gamma}w(n)$, $Q(n)=\sum_{i=0}^{m-1}\theta_i n^{1-i/m}$.
5. *When $s>0$, we have $\sigma=-s/m$ or $q=-s$.*
   In this case, $a_n=[\Gamma(n)]^{s/m}\exp[Q(n)]n^{\gamma}w(n)$, $Q(n)=\sum_{i=0}^{m-1}\theta_i n^{1-i/m}$.

Of course, $w(n)$ is in $\tilde{\mathbf{A}}_0^{(0,m)}$ in all cases.

We now restrict our attention to the cases described in parts 1–4 of Theorem 2.5, for which the series $\sum_{n=1}^{\infty}a_n$ (i) either converges (ii) or diverges but has an Abel sum or an Hadamard finite part that serves as the antilimit of $A_n=\sum_{k=1}^{n}a_k$ as $n\to\infty$. (In part 5, the series $\sum_{n=1}^{\infty}a_n$ always diverges and has no Abel sum or Hadamard finite part. It may have a Borel sum, however.)

1. In part 1, we assume that $\gamma\neq -1+i/m$, $i=0,1,\ldots,$ as in part of Theorem 2.2. We have two cases to consider:
   - If $\Re\gamma<-1$, $\sum_{n=1}^{\infty}a_n$ converges.
   - If $\Re\gamma\geq -1$, $\sum_{n=1}^{\infty}a_n$ diverges but has an Hadamard finite part that serves as the antilimit of $A_n=\sum_{k=1}^{n}a_k$ as $n\to\infty$.
2. In part 2, we assume the following two situations:
   - $\lim_{n\to\infty}\Re Q(n)=-\infty$ or, equivalently, $\Re\theta_r<0$. In this case, $\sum_{n=1}^{\infty}a_n$ converges for all γ. [If $\Re\theta_r>0$, then $\lim_{n\to\infty}\Re Q(n)=+\infty$; therefore, $\sum_{n=1}^{\infty}a_n$ diverges for all γ.]
   - $\Re Q(n)=0$ or, equivalently, $\Re\theta_i=0$, $i=r,\ldots,m-1$.
     - $\sum_{n=1}^{\infty}a_n$ converges if $\Re\gamma<-r/m$.
     - $\sum_{n=1}^{\infty}a_n$ diverges if $\Re\gamma\geq -r/m$ but has an Abel sum that serves as the antilimit of $A_n=\sum_{k=1}^{n}a_k$ as $n\to\infty$.
3. In part 3, as in item 2, we assume the following two situations:

[5] Thus, $\{a_n\}\notin\tilde{\mathbf{b}}^{(m)}$ if $c(n)=a_{n+1}/a_n$ is in $\tilde{\mathbf{A}}_0^{(0,m)}$ and has an asymptotic expansion of the form $c(n)\sim 1+\sum_{i=m+1}^{\infty}c_i n^{-i/m}$ as $n\to\infty$.

- $\lim_{n\to\infty} \Re Q(n) = -\infty$ or, equivalently, $\Re\theta_0 < 0$, which is equivalent to $|c_0| < 1$. In this case, $\sum_{n=1}^{\infty} a_n$ converges for all γ. (If $\Re\theta_0 > 0$, which is equivalent to $|c_0| > 1$, $\sum_{n=1}^{\infty} a_n$ diverges for all γ.)
- $\Re Q(n) = 0$ or, equivalently, $\Re\theta_i = 0$, $i = 0, 1, \ldots, m-1$. (Note that we now have $|c_0| = 1$, in addition to $c_0 \neq 1$.) We now have the following cases:
  - $\sum_{n=1}^{\infty} a_n$ converges if $\Re\gamma < 0$.
  - $\sum_{n=1}^{\infty} a_n$ diverges if $\Re\gamma \geq 0$ but has an Abel sum that serves as the antilimit of $A_n = \sum_{k=1}^{n} a_k$ as $n \to \infty$.

4. In part 4, we do not assume anything in addition to what is there. In this case, $\sum_{n=1}^{\infty} a_n$ converges for all γ.

Remark 2 Note that in all the cases considered above, we have $\sigma = q/m$, with $q \in \{0, 1, \ldots, m\}$.

Theorem 2.6 that follows concerns the summation properties of sequences $\{a_n\}$ in $\tilde{\mathbf{b}}^{(m)}$ and is the most important result that we use in developing the $\tilde{d}^{(m)}$ transformation. Its proof relies on Theorems 2.2, 2.4, and 2.5 and is quite involved. We continue to use the notation of Theorem 2.5 and $A_n = \sum_{k=1}^{n} a_k$.

Theorem 2.6 ([18], **Theorem 6.6.6**) *Let $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$ for which the infinite series $\sum_{n=1}^{\infty} a_n$ converges or diverges but has an Abel sum or Hadamard finite part. Then there exist a constant S and a function $g \in \tilde{\mathbf{A}}_0^{(0,m)}$ strictly such that*

$$A_{n-1} = S + n^{\sigma} a_n\, g(n), \tag{2.9}$$

whether $\sum_{n=1}^{\infty} a_n$ converges or not. Here, S is the sum of $\sum_{n=1}^{\infty} a_n$ when the latter converges; otherwise, S is the Abel sum or the Hadamard finite part of $\sum_{n=1}^{\infty} a_n$.

Remark 3 Before closing, we would like to mention that we can use the $\tilde{d}^{(m)}$ transformation for computing the sums of the two (trigonometric-type) series $S^{(c)} = \sum_{n=1}^{\infty} a_n^{(c)}$ and $S^{(s)} = \sum_{n=1}^{\infty} a_n^{(s)}$ with

$$a_n^{(c)} = [\Gamma(n)]^{s/m} e^{u(n)} \cos(v(n)) h(n) \quad \text{and} \quad a_n^{(s)} = [\Gamma(n)]^{s/m} e^{u(n)} \sin(v(n)) h(n),$$

where $u(n)$ and $v(n)$ are *real* polynomials of degree at most m in $n^{1/m}$ and $h(n) \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$ is not necessarily real. Clearly, neither of the sequences $\{a_n^{(c)}\}$ or $\{a_n^{(s)}\}$ belongs to $\tilde{\mathbf{b}}^{(m)}$. The two sequences $\{a_n^{(\pm)}\}$, where

$$a_n^{(\pm)} = [\Gamma(n)]^{s/m} \exp[Q^{(\pm)}(n)] h(n); \quad Q^{(\pm)}(n) = u(n) \pm \mathrm{i} v(n),$$

do belong to $\tilde{\mathbf{b}}^{(m)}$, however. In view of this observation, we can now apply the $\tilde{d}^{(m)}$ transformation to the two series $S^{(\pm)} = \sum_{n=1}^{\infty} a_n^{(\pm)}$ successfully. Clearly,

$$S^{(c)} = \frac{S^{(+)} + S^{(-)}}{2} \quad \text{and} \quad S^{(s)} = \frac{S^{(+)} - S^{(-)}}{2\mathrm{i}}.$$

In case $h(n)$ is real, it is sufficient to apply the $\tilde{d}^{(m)}$ transformation to $S^{(+)}$ only since, in this case,

$$S^{(c)} = \Re S^{(+)} \quad \text{and} \quad S^{(s)} = \Im S^{(+)}.$$

3 The $\tilde{d}^{(m)}$ transformation

3.1 Derivation of the $\tilde{d}^{(m)}$ transformation

Consider now the cases in which Theorem 2.6 applies and (2.9) holds. Being in $\tilde{\mathbf{A}}_0^{(0,m)}$ strictly, the function $g(n)$ in (2.9) has the asymptotic expansion

$$g(n) \sim \sum_{i=0}^{\infty} g_i n^{-i/m} \quad \text{as } n \to \infty, \quad g_0 \neq 0. \tag{3.1}$$

Consequently, (2.9) can be expressed as in

$$A_{n-1} \sim S + n^{\sigma} a_n \sum_{i=0}^{\infty} g_i n^{-i/m} \quad \text{as } n \to \infty, \quad g_0 \neq 0. \tag{3.2}$$

We now go on to the development of the $\tilde{d}^{(m)}$ transformation: First, we truncate the infinite summation in (3.2) at the term $i = n-1$, replace the asymptotic equality sign $\sim$ by the equality sign $=$, and replace S by $\tilde{d}_n^{(m,j)}$ and the β_i by $\bar{\beta}_i$. Next, we choose positive integers R_l, $l = 0, 1, \ldots,$ that are ordered as in

$$1 \leq R_0 < R_1 < R_2 < \cdots, \tag{3.3}$$

and we set up the $(n+1) \times (n+1)$ system of linear equations

$$A_{R_l-1} = \tilde{d}_n^{(m,j)} + R_l^{\hat{\sigma}} a_{R_l} \sum_{i=0}^{n-1} \frac{\bar{\beta}_i}{(R_l+\alpha)^{i/m}}, \quad j \leq l \leq j+n, \tag{3.4}$$

where $\hat{\sigma} = \sigma$ when σ is known or $\hat{\sigma}$ is a known upper bound for σ. (Needless to say, if we know the exact value of σ, especially $\sigma = 0$, we should use it. Since $\sigma \leq m/m = 1$ in all cases, we can always choose $\hat{\sigma} = 1$ and be sure that the $\tilde{d}^{(m)}$ transformation will accelerate convergence in all cases.)[6] Here, $\alpha > -R_0$ and a good choice in many cases is $\alpha = 0$. As can be seen from (3.4), to compute $\tilde{d}_n^{(m,j)}$, we need the first R_{j+n} terms of the infinite series, namely, $a_1, a_2, \ldots, a_{R_{j+n}}$.

Note that the unknowns in (3.5) are $\tilde{d}_n^{(m,j)}$ and $\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_{n-1}$. Of these, $\tilde{d}_n^{(m,j)}$ is the approximation to S and the $\bar{\beta}_i$ are additional auxiliary unknowns. We call this procedure the $\tilde{d}^{(m)}$ transformation. This transformation is actually a generalized Richardson extrapolation method in the class GREP$^{(1)}$, which is the simplest prototype of the *generalized Richardson extrapolation procedure* GREP$^{(m)}$ of the author [13]; see also Sidi [18, Chapters 4–7].

[6] The choice $\hat{\sigma} = 1$ results in the "universal" formulation of the $\tilde{d}^{(m)}$ transformation that is applicable in the presence of all $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$ that we mentioned in the paragraph preceding the last in Section 1.

The approximations $\tilde{d}_n^{(m,j)} \equiv A_n^{(j)}$ can be arranged in a two-dimensional array as in Table 1. Note that $\tilde{d}_0^{(m,j)} = A_0^{(j)} = A_{R_{j-1}}, j = 0, 1, \ldots$.

When $\hat{\sigma} = \hat{q}/m \geq 0$, where $\hat{q}$ is a nonnegative integer, the equations in (3.4) can be replaced by

$$A_{R_l} = \tilde{d}_n^{(m,j)} + R_l^{\hat{\sigma}} a_{R_l} \sum_{i=0}^{n-1} \frac{\bar{\beta}_i}{(R_l + \alpha)^{i/m}}, \quad j \leq l \leq j+n, \tag{3.5}$$

the solution for $\tilde{d}_n^{(m,j)}$ remaining the same as in (3.4). This amounts to adding a_{R_l} to both sides of (3.4), and replacing $\bar{\beta}_{\hat{q}}$ by $\bar{\beta}_{\hat{q}} + 1$. In our numerical examples, we have taken $\hat{\sigma} = m/m = 1$ and used (3.5) to define $\tilde{d}_n^{(m,j)}$. Note that now $A_0^{(j)} = A_{R_j}, \ j = 0, 1, \ldots,$ in Table 1. Note also that, with $\hat{\sigma} = 1$ in (3.5), we do not need any further information about $Q(n)$ and the parameters s, r, and γ in (1.1); mere knowledge of the fact that $\{a_n\}$ is in $\tilde{\mathbf{b}}^{(m)}$ is sufficient for applying the $\tilde{d}^{(m)}$ transformation successfully.

Looking at how the approximations $A_n^{(j)}$ are placed in Table 1, we call the sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ (with n fixed) *column sequences*. Similarly, we call the sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ (with j fixed) *diagonal sequences*. The known theoretical results and numerical experience suggest that diagonal sequences have superior convergence properties and are much better than column sequences when the latter converge. Furthermore, numerical experience suggests that diagonal sequences converge to some antilimit when the series $\sum_{n=1}^{\infty} a_n$ diverges. This can be proved rigorously at least in some cases. Normally, we look at the diagonal sequence $\{A_n^{(0)}\}_{n=0}^{\infty}$.

We review some of the convergence theory pertaining to the $\tilde{d}^{(m)}$ transformation in Section 3.5.

3.2 Assessing the numerical stability of the $\tilde{d}^{(m)}$ transformation

An important issue that is critical at times when computing the approximations $\tilde{d}_n^{(m,j)}$ is that of numerical stability in the presence of finite-precision arithmetic. We show how this can be tackled effectively next.

Table 1 Approximations $\tilde{d}_n^{(m,j)} \equiv A_n^{(j)}$ arranged in a two-dimensional array

$A_0^{(0)}$				
$A_0^{(1)}$	$A_1^{(0)}$			
$A_0^{(2)}$	$A_1^{(1)}$	$A_2^{(0)}$		
$A_0^{(3)}$	$A_1^{(2)}$	$A_2^{(1)}$	$A_3^{(0)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

By Cramer's rule on the linear system in (3.5), $\tilde{d}_n^{(m,j)}$ can be expressed in the form

$$\tilde{d}_n^{(m,j)} = \sum_{i=0}^{n} \gamma_{n,i}^{(j)} A_{R_{j+i}} \equiv A_n^{(j)}, \tag{3.6}$$

with some scalars $\gamma_{n,0}^{(j)}, \gamma_{n,1}^{(j)}, \dots, \gamma_{n,n}^{(j)}$ that satisfy $\sum_{i=0}^{n} \gamma_{n,i}^{(j)} = 1$. As discussed in [18], the numerical stability of the $A_n^{(j)}$ computed in finite-precision arithmetic can be assessed reliably as follows: Denote the numerically computed A_i and $A_n^{(j)}$ by $\bar{A}_i$ and $\bar{A}_n^{(j)}$, respectively. Then $\bar{A}_n^{(j)} - S$, the actual numerical error in $\bar{A}_n^{(j)}$, satisfies

$$|\bar{A}_n^{(j)} - S| \le |\bar{A}_n^{(j)} - A_n^{(j)}| + |A_n^{(j)} - S|. \tag{3.7}$$

The term $|A_n^{(j)} - S|$ is the exact (theoretical) error in $A_n^{(j)}$ and, assuming convergence, it tends to zero as $j \to \infty$ or as $n \to \infty$. The term $|\bar{A}_n^{(j)} - A_n^{(j)}|$, however, remains a positive quantity, meaning that the computational error $|\bar{A}_n^{(j)} - A_n^{(j)}|$ dominates the actual error in $\bar{A}_n^{(j)}$; that is,

$$|\bar{A}_n^{(j)} - S| \approx |\bar{A}_n^{(j)} - A_n^{(j)}| \quad \text{for large } j \text{ or } n. \tag{3.8}$$

We now consider two different but related approaches to the estimation of $|\bar{A}_n^{(j)} - A_n^{(j)}|$, hence to the estimation of the numerical stability:

1. Let us denote the absolute error in the computation of A_i by ϵ_i; thus, $\bar{A}_i = A_i + \epsilon_i$. Then, assuming that the computed $\gamma_{n,i}^{(j)}$ are not much different from the exact ones,[7] we have

$$\bar{A}_n^{(j)} \approx \sum_{i=0}^{n} \gamma_{n,i}^{(j)} \bar{A}_{R_{j+i}} = A_n^{(j)} + \sum_{i=0}^{n} \gamma_{n,i}^{(j)} \epsilon_{R_{j+i}},$$

from which, we obtain

$$|\bar{A}_n^{(j)} - A_n^{(j)}| \lessapprox \Gamma_n^{(j)} \left(\max_{0 \le i \le n} |\epsilon_{R_{j+i}}| \right), \tag{3.9}$$

where

$$\Gamma_n^{(j)} = \sum_{i=0}^{n} |\gamma_{n,i}^{(j)}| \ge 1. \tag{3.10}$$

Consequently, in case of convergence, (3.8) becomes

$$|\bar{A}_n^{(j)} - S| \lessapprox \Gamma_n^{(j)} \left(\max_{0 \le i \le n} |\epsilon_{R_{j+i}}| \right), \quad \text{for large } j \text{ or } n. \tag{3.11}$$

If the A_i are computed with machine accuracy and the roundoff unit of the floating-point arithmetic being used is $\mathbf{u}$, then we have $|\epsilon_i| \le |A_i|\mathbf{u}$. In case the

[7] The explanation for this is twofold: (i) Numerical computations show this. (ii) What matters is not so much the exact value of $\Gamma_n^{(j)}$ in (3.9)–(3.12) and of $\Lambda_n^{(j)}$ in (3.14)–(3.18), but rather their orders of magnitude, as explained following (3.18) and as many numerical examples show very clearly.

series $\sum_{n=1}^{\infty} a_n$ converges, we have that the A_i are approximately equal to, or of the same order as, S. Therefore, (3.11) can be replaced by

$$\frac{|\bar{A}_n^{(j)} - S|}{|S|} \lessapprox \Gamma_n^{(j)}\mathbf{u}, \quad \text{for large } j \text{ or } n. \tag{3.12}$$

In such a case, if $\Gamma_n^{(j)}\mathbf{u} = O(10^{-p})$, where p is a positive integer, then the relative error in $\bar{A}_n^{(j)}$ is $O(10^{-p})$, that is, we can rely on p of the significant figures of $\bar{A}_n^{(j)}$ as being correct for j or n large.

Finally, by Theorem 7.2.3 in [18, p. 161],

$$\Gamma_n^{(j)} = 1 \quad \text{if } a_n a_{n+1} < 0, \quad n = 1, 2, \ldots. \tag{3.13}$$

2. Let us denote the relative error in the computation of A_i by η_i; thus $\bar{A}_i = A_i(1+\eta_i)$. Then, assuming again that the computed $\gamma_{n,i}^{(j)}$ are not much different from the exact ones, we have

$$\bar{A}_n^{(j)} \approx \sum_{i=0}^{n} \gamma_{n,i}^{(j)} \bar{A}_{R_{j+i}} = A_n^{(j)} + \sum_{i=0}^{n} \gamma_{n,i}^{(j)} A_{R_{j+i}} \eta_{R_{j+i}},$$

from which, we obtain

$$|\bar{A}_n^{(j)} - A_n^{(j)}| \lessapprox \Lambda_n^{(j)} \left(\max_{0 \le i \le n} |\eta_{R_{j+i}}| \right), \tag{3.14}$$

where

$$\Lambda_n^{(j)} = \sum_{i=0}^{n} |\gamma_{n,i}^{(j)}|\,|A_{R_{j+i}}|. \tag{3.15}$$

Consequently, in case of convergence, (3.8) becomes

$$|\bar{A}_n^{(j)} - S| \lessapprox \Lambda_n^{(j)} \left(\max_{0 \le i \le n} |\eta_{R_{j+i}}| \right), \quad \text{for large } j \text{ or } n. \tag{3.16}$$

The bound in (3.16) is especially useful when $\{A_n\}$ is a divergent sequence (that is, when $\sum_{n=1}^{\infty} a_n$ diverges) but the antilimit S of $\{A_n\}$ exists and $A_n^{(j)} \to S$ as $j \to \infty$ or $n \to \infty$.[8]

If the A_i are computed with machine accuracy, then we have $|\eta_i| \le \mathbf{u}$, where $\mathbf{u}$ is the roundoff unit of the floating-point arithmetic being used. In such a case, we have

$$|\bar{A}_n^{(j)} - S| \lessapprox \Lambda_n^{(j)}\mathbf{u}, \quad \text{for large } j \text{ or } n. \tag{3.17}$$

If we want to assess the relative error in $\bar{A}_n^{(j)}$, we simply divide the right-hand side of (3.17) by $\bar{A}_n^{(j)}$, obtaining

$$\frac{|\bar{A}_n^{(j)} - S|}{|S|} \lessapprox \frac{\Lambda_n^{(j)}}{|\bar{A}_n^{(j)}|}\mathbf{u}, \quad \text{for large } j \text{ or } n, \tag{3.18}$$

[8] It is clear that (3.12) is useless when $\{A_n\}$ diverges.

as an estimate of the relative error in $\bar{A}_n^{(j)}$. If $|\Lambda_n^{(j)}/\bar{A}_n^{(j)}|\mathbf{u} = O(10^{-p})$ for some positive integer p, then we can conclude that, as an approximation to S, $\bar{A}_n^{(j)}$ has approximately p correct significant figures, close to convergence. Surprisingly, this seems to be the case also when the series $\sum_{n=1}^{\infty} a_n$ diverges weakly or strongly.

Let us assume that the exact/theoretical diagonal sequence of approximations $\{A_n^{(0)}\}_{n=0}^{\infty}$ is converging to the limit or antilimit of the sequence $\{A_n\}_{n=1}^{\infty}$. From our discussion above, the following conclusion can be reached concerning the numerically computed diagonal sequence of approximations $\{\bar{A}_n^{(0)}\}_{n=0}^{\infty}$: If the corresponding sequences $\{\Gamma_n^{(0)}\}_{n=0}^{\infty}$ and/or $\{\Lambda_n^{(0)}\}_{n=0}^{\infty}$ are *increasing* quickly, then the accuracy of $\{\bar{A}_n^{(0)}\}_{n=0}^{\infty}$ is *decreasing* quickly, by (3.12) and/or (3.18). Thus, $\bar{A}_n^{(0)}$ may be improving (gaining more and more correct significant digits) for $n = 0, 1, \dots, N$, for some N, and it deteriorates for $n = N+1, N+2, \dots$, in the sense that it eventually loses all of its correct significant digits; that is, adding more terms of the series $\sum_{n=1}^{\infty} a_n$ in the extrapolation process does not help to improve the approximations $\bar{A}_n^{(0)}$. This is how numerical instability exhibits itself.

In Section 3.4, we shall show how the $\Gamma_n^{(j)}$ and $\Lambda_n^{(j)}$ can be computed recursively and without having to know anything other than the sequence $\{a_n\}$.

3.3 Choice of the R_l

As is obvious from (3.12) and (3.18), the smaller $\Gamma_n^{(j)}$ and/or $\Lambda_n^{(j)}$, the better the numerical stability, hence the accuracy, of the $A_n^{(j)}$. This can be achieved by picking the integers R_l in (3.4) and (3.5) in one of the following two forms:

1. Pick real numbers $\kappa \geq 1$ and $\eta \geq 1$ and set

$$R_l = \lfloor \kappa l + \eta \rfloor, \quad l = 0, 1, \dots. \tag{3.19}$$

   We call this choice of the R_l *arithmetic progression sampling* and denote it by APS for short. Clearly, $\lim_{l\to\infty} R_l/l = \kappa$, which implies that $R_l \sim \kappa l$ as $l \to \infty$, hence $\lim_{l\to\infty} R_l/R_{l-1} = 1$. Note also that

$$\kappa - 1 < R_l - R_{l-1} < \kappa + 1 \Rightarrow |(R_l - R_{l-1}) - \kappa| < 1, \forall\, l \geq 1, \tag{3.20}$$

   whether κ is an integer or not. Of course, the simplest APS is one in which $\kappa = 1$ and $\eta = 1$, that is, $R_l = l + 1, l = 0, 1, \dots$.
2. Pick a real number $\tau > 1$ and set

$$R_0 = 1; \quad R_l = \max\{\lfloor \tau R_{l-1} \rfloor, l+1\}, \quad l = 1, 2, \dots. \tag{3.21}$$

   We call this choice of the R_l *geometric progression sampling* and denote it by GPS for short. In this case, we have (see Sidi [19, Section 3.4])

$$R_l = \begin{cases} l+1, & l = 0, 1, \dots, L-1, \\ \lfloor \tau R_{l-1} \rfloor, & l = L, L+1, L+2, \dots, \end{cases} \tag{3.22}$$

where

$$L = \left\lceil \frac{2}{\tau - 1} \right\rceil. \tag{3.23}$$

In addition, $\lim_{l\to\infty} R_l/R_{l-1} = \tau$, which implies that R_l increases as τ^l. Indeed, GPS generates a sequence of integers R_l that satisfy $b_1\tau^l \le R_l \le b_2\tau^l$ for some positive constants $b_1 \le b_2$, hence grow exponentially precisely like τ^l. When τ is an integer ≥ 2, then $R_l = \tau^l$ for all $l \ge 0$. Of course, we do not want R_l to increase very fast as this means that we need a lot of the terms of the series $\sum_{n=1}^{\infty} a_n$ in applying the $\tilde{d}^{(m)}$ transformation; therefore, we take $1 < \tau < 2$, for example.

Remark 4 Note that the sequence of the integers R_l generated by APS with noninteger κ is very closely an arithmetic sequence, while that generated by GPS with noninteger τ is very closely a geometric sequence.

In essentially the same form described here, APS (with integer κ and η) and GPS were originally suggested in Ford and Sidi [8, Appendix B]. For a detailed discussion of the subject, see Sidi [18, Chapter 10].

3.4 Recursive implementation via the W-algorithm

The W-algorithm of Sidi [14] and its extensions in [15] and [18, Section 7.2] can be used to implement GREP$^{(1)}$ and study its numerical stability very efficiently. Specifically, the approximations $\tilde{d}_n^{(m,j)}$ (with $\alpha = 0$ in (3.5)) and the $\Gamma_n^{(j)}$ and the $\Lambda_n^{(j)}$, which are the quantities developed for assessing the numerical stability of the $\tilde{d}_n^{(m,j)}$, can be computed very economically, *and without having to determine either the* $\bar{\beta}_i$ *in* (3.5) *or the* $\gamma_{n,i}^{(j)}$ *in* (3.6)*,* as follows:

1. For $j = 0, 1, \ldots,$ compute

$$M_0^{(j)} = \frac{A_{R_j}}{\omega_{R_j}}, \quad N_0^{(j)} = \frac{1}{\omega_{R_j}}; \qquad \omega_r \equiv r^{\hat{\sigma}} a_r,$$
$$H_0^{(j)} = (-1)^j |N_0^{(j)}|, \quad K_0^{(j)} = (-1)^j |M_0^{(j)}|.$$

2. For $j = 0, 1, \ldots,$ and $n = 1, 2 \ldots,$ compute

$$M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{R_{j+n}^{-1/m} - R_j^{-1/m}}, \quad N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{R_{j+n}^{-1/m} - R_j^{-1/m}},$$
$$H_n^{(j)} = \frac{H_{n-1}^{(j+1)} - H_{n-1}^{(j)}}{R_{j+n}^{-1/m} - R_j^{-1/m}}, \quad K_n^{(j)} = \frac{K_{n-1}^{(j+1)} - K_{n-1}^{(j)}}{R_{j+n}^{-1/m} - R_j^{-1/m}}.$$

3. For $j = 0, 1, \ldots,$ and $n = 1, 2 \ldots,$ compute

$$A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}} \equiv \tilde{d}_n^{(m,j)}, \quad \Gamma_n^{(j)} = \left|\frac{H_n^{(j)}}{N_n^{(j)}}\right|, \quad \Lambda_n^{(j)} = \left|\frac{K_n^{(j)}}{N_n^{(j)}}\right|.$$

Of course, the $M_n^{(j)}$, $N_n^{(j)}$, $H_n^{(j)}$, and $K_n^{(j)}$ can be arranged in separate two-dimensional tables just like the $A_n^{(j)}$ in Table 1. For details, see [18, Section 7.2].

Here, we have taken $\hat{\sigma} \geq 0$ and used the definition given in (3.5); hence, $M_0^{(j)} = A_{R_j}/\omega_{R_j}$. If $\hat{\sigma} < 0$, then we should use the definition given in (3.4); therefore, $M_0^{(j)}$ should now be computed as $M_0^{(j)} = A_{R_j-1}/\omega_{R_j}$.

Note that the input needed for computing $\Gamma_n^{(j)}$ and $\Lambda_n^{(j)}$ is precisely that used to compute $A_n^{(j)}$; nothing else is needed.

3.5 Some convergence results for the $\tilde{d}^{(m)}$ transformation

As already mentioned, the $\tilde{d}^{(m)}$ transformation is a GREP$^{(1)}$, and the convergence properties of GREP$^{(1)}$ are studied in detail in Sidi [15–17], and [18, Chapters 8, 9]. Powerful results on the convergence and stability of the $\tilde{d}^{(m)}$ transformation, as it is being applied to the cases in which $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$, can thus be found in Sidi [18, Chapters 8, 9]:

- For the case $a_n = n^\gamma w(n)$, $w \in \tilde{\mathbf{A}}_0^{(0,m)}$, that is, $s = 0$ and $Q(n) \equiv 0$, (mentioned in [18, Example 8.2.3]), see the theorems in [18, Chapter 8].
- For the cases $a_n = e^{Q(n)} n^\gamma w(n)$ or $a_n = [\Gamma(n)]^{s/m} n^\gamma w(n)$ or $a_n = [\Gamma(n)]^{s/m} e^{Q(n)} n^\gamma w(n)$, $w \in \tilde{\mathbf{A}}_0^{(0,m)}$, (mentioned in [18, Example 9.2.3]), see the theorems in [18, Chapter 9].

Below, we state some convergence theorems that follow from those in [18]. Here, we are assuming that the functions $\mu(t) \equiv w(t^{-m})$ and $B(t) \equiv g(t^{-m})$ are both infinitely differentiable as functions of t in some interval $[0, \hat{t}]$, $\hat{t} > 0$. The function $g(n)$ is the one that appears in Theorem 2.6.

Theorem 3.1 *Let $a_n = n^\gamma w(n)$ with $w \in \tilde{\mathbf{A}}_0^{(0,m)}$. Then, the following are true:*

1. *The column sequences $\{A_n^{(j)}\}_{j=0}^\infty$ (with fixed n) obtained with both APS and GPS satisfy*

$$A_n^{(j)} - S = O(R_j\, a_{R_j} R_j^{-n/m}) = O(R_j^{\gamma+1-n/m}) \quad as\ j \to \infty.$$

   In addition, $\lim_{j\to\infty} \Gamma_n^{(j)} = \infty$ for APS, and $\lim_{j\to\infty} \Gamma_n^{(j)} < \infty$ for GPS.[9]

2. *When γ is real, the diagonal sequences $\{A_n^{(j)}\}_{n=0}^\infty$ (with fixed j) obtained with GPS converge to S whether $\sum_{k=1}^\infty a_k$ converges or not. We actually have*

$$A_n^{(j)} - S = O(e^{-\lambda n}) \quad as\ n \to \infty \quad \forall\, \lambda > 0.$$

[9]Thus, $\lim_{j\to\infty} A_n^{(j)} = S$ (i) for all $n \geq 1$ if $\sum_{k=1}^\infty a_k$ converges, that is, if $\Re\gamma < -1$, and (ii) for $n > m(\Re\gamma + 1)$ if $\sum_{k=1}^\infty a_k$ diverges, that is, if $\Re\gamma \geq -1$, in which case, S is the antilimit.

This result holds also when γ is complex, with $R_l = \tau^l$, τ being an integer[10].

Theorem 3.2 *Let $a_n = e^{Q(n)} n^{\gamma} w(n)$, with $Q(n) = \sum_{i=0}^{m-1} \theta_i n^{1-i/m}$, such that $\theta_0 \neq 0$ and $\lim_{n\to\infty} \Re Q(n) \neq +\infty$.*[11] *Choose R_l via APS as $R_l = \kappa(l+1)$, κ an integer. Then, the following are true:*

1. *Provided $e^{\kappa\theta_0} \neq 1$,*[12] *the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ (with fixed n) satisfy*

$$A_n^{(j)} - S = O(R_j\, a_{R_j} R_j^{-n/m} j^{-n}) = O(R_j\, a_{R_j} j^{-n/m-n}) \quad \text{as } j \to \infty.$$

In addition, $\lim_{j\to\infty} \Gamma_n^{(j)} < \infty$.

2. *Assume a_n is real and of the form $a_n = (-1)^n e^{\tilde{Q}(n)} n^{\gamma} w(n)$, that is, $Q(n) = \mathrm{i}\pi n + \sum_{i=0}^{m-1} \tilde{\theta}_i n^{1-i/m}$, $\tilde{\theta}_i$ real. Then, whether $\sum_{k=1}^{\infty} a_k$ converges or not, the diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ (with fixed j) obtained via APS, with $R_l = l+1$, converge to S. We actually have*

$$A_n^{(j)} - S = O(e^{-\lambda n}) \quad \text{as } n \to \infty \quad \forall\, \lambda > 0.$$

In addition, $\Gamma_n^{(j)} = 1$.

Remark 5 1. Note that, in both theorems, $A_{R_j} - A = O(R_j^{\sigma} a_{R_j})$ as $j \to \infty$ by Theorem 2.6; thus our results in part 1 of both theorems show clearly that convergence acceleration is taking place as $j \to \infty$ and also give precise quantifications of the acceleration.

2. In part 2 of both theorems, $A_n^{(j)} - S$ tends to zero as $n \to \infty$ faster than *any* exponential function $e^{-\lambda n}$ with $\lambda > 0$. It is thus clear that both theorems show that convergence acceleration is taking place as $n \to \infty$.

4 Illustrative examples for Theorem 2.6

We now verify Theorem 2.6 in the form given in (3.1) with a few examples, to which we will return later in Section 5. The examples we choose are different kinds of telescoping series, both convergent and divergent, in which the limits or the antilimits are identified immediately. In these examples, we have two types of series:

[10] At the present, we do not have a theorem that covers cases with complex γ when GPS is used with noninteger τ in (3.21).

[11] This means that a_n tends to zero exponentially or behaves at worst like a fixed power of n as $n \to \infty$.

[12] Note that $e^{\kappa\theta_0} = 1$ only when θ_0 is purely imaginary and $\kappa|\theta_0|$ is an integer multiple of 2π.

$$\text{Type 1:} \quad a_n = \delta_n - \delta_{n-1} \Rightarrow A_n = \sum_{k=1}^{n} a_k = -\delta_0 + \delta_n, \tag{4.1}$$

$$\text{Type 2:} \quad a_n = (-1)^n(\delta_n + \delta_{n-1}) \Rightarrow A_n = \sum_{k=1}^{n} a_k = -\delta_0 + (-1)^n\delta_n. \tag{4.2}$$

Remark 6 It seems to be quite difficult to find infinite series $\sum_{n=1}^{\infty} a_n$ with simple $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$ with known sums. (Our efforts to find such series in the literature have not produced any positive result.) In view of this limitation, the examples we construct here as type 1 and type 2 series are very appropriate. As we will see shortly in Section 5, their limits or antilimits are simply $-\delta_0$, which are known quantities.

For simplicity, let us now take [see Theorem 2.4]

$$\delta_n = (n!)^{s/m} e^{Q(n)}; \quad Q(n) = \theta_0 n + \sum_{i=r}^{m-1} \theta_i n^{1-i/m},$$
$$s \text{ integer}, \quad \theta_0 \text{ real}, \quad \theta_r \neq 0, \quad r \in \{1, \ldots, m-1\}. \tag{4.3}$$

Clearly,

- when $s = 0$, $\sum_{n=1}^{\infty} a_n$ is convergent if $\lim_{n\to\infty} \Re Q(n) = -\infty$ and divergent if $\lim_{n\to\infty} \Re Q(n) = +\infty$,
- when $s < 0$, $\sum_{n=1}^{\infty} a_n$ is always convergent, and
- when $s > 0$, $\sum_{n=1}^{\infty} a_n$ is always divergent.

We now would like to verify/confirm that, for both types of series, the sequences $\{a_n\}$ are in $\tilde{\mathbf{b}}^{(m)}$; that is, (i) the relevant a_n are precisely of the form given in (1.1)–(1.2) and (ii) the partial sums $A_n = \sum_{i=1}^{n} a_i$ satisfy (2.9) in Theorem 2.6.

4.1 Analysis of a_n

We now analyze, in a unified manner, the asymptotic behavior of a_n as $n \to \infty$, recalling (4.1) and (4.2). By the fact that

$$\delta_n/\delta_{n-1} = n^{s/m} e^{f(n)}; \quad f(n) = Q(n) - Q(n-1) = \Delta Q(n-1), \tag{4.4}$$

we have

$$\text{Type 1:} \quad a_n = \delta_{n-1} q(n); \qquad q(n) = \delta_n/\delta_{n-1} - 1 = n^{s/m} e^{f(n)} - 1, \tag{4.5}$$
$$\text{Type 2:} \quad a_n = (-1)^n \delta_{n-1} q(n); \qquad q(n) = \delta_n/\delta_{n-1} + 1 = n^{s/m} e^{f(n)} + 1. \tag{4.6}$$

Now, by the binomial theorem, for $n \geq 2$,

$$\Delta(n-1)^p = n^p - (n-1)^p = n^p[1 - (1 - n^{-1})^p] = \sum_{j=1}^{\infty} (-1)^{j+1} \binom{p}{j} n^{p-j},$$

from which we have

$$\Delta(n-1)^p = pn^{p-1} - \frac{p(p-1)}{2!}n^{p-2} + O(n^{p-3}) \quad \text{as } n \to \infty.$$

Thus,

$$\begin{aligned} f(n) &= \theta_0 + \sum_{i=r}^{m-1} \theta_i \Delta(n-1)^{1-i/m} \\ &= \theta_0 + \sum_{i=r}^{m-1} \theta_i [(1-i/m)n^{-i/m} + O(n^{-1-i/m})] \quad \text{as } n \to \infty, \\ &= \theta_0 + \sum_{i=r}^{m-1} \theta_i (1-i/m)n^{-i/m} + O(n^{-1-r/m}) \quad \text{as } n \to \infty, \end{aligned}$$

and this gives

$$f(n) = \theta_0 + \sum_{i=r}^{m-1} \theta_i (1-i/m)n^{-i/m} + u(n), \quad u(n) \in \tilde{\mathbf{A}}_0^{(-1-r/m,m)}. \tag{4.7}$$

Consequently,

$$\begin{aligned} e^{f(n)} &= c_0 \cdot \exp\left[\sum_{i=r}^{m-1} \theta_i (1-i/m)n^{-i/m} + u(n)\right]; \quad c_0 = e^{\theta_0} > 0, \\ &= c_0[1 + \theta_r(1-r/m)n^{-r/m} + v(n)], \quad v(n) \in \tilde{\mathbf{A}}_0^{(-(r+1)/m,m)}, \\ &= c_0[1 + h(n)], \quad h(n) \in \tilde{\mathbf{A}}_0^{(-r/m,m)} \text{ strictly, because } \theta_r \neq 0. \end{aligned} \tag{4.8}$$

We now make use of this in the analysis of a_n in the two types of series:

1. Type 1: By (4.5) and (4.8), we have $a_n = \delta_{n-1} q(n)$, where

$$q(n) = c_0 n^{s/m}[1 + h(n)] - 1, \quad h(n) \in \tilde{\mathbf{A}}_0^{(-r/m,m)} \text{ strictly.} \tag{4.9}$$

Then we have the following:

$$q(n) = h(n) \in \tilde{\mathbf{A}}_0^{(-r/m,m)} \text{ strictly, when } s = 0 \text{ and } \theta_0 = 0. \tag{4.10}$$

$$q(n) = (c_0 - 1) + c_0 h(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \text{ strictly, when } s = 0 \text{ and } \theta_0 \neq 0. \tag{4.11}$$

$$q(n) \in \tilde{\mathbf{A}}_0^{(s/m,m)} \text{ strictly, when } s > 0. \tag{4.12}$$

$$q(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \text{ strictly, when } s < 0. \tag{4.13}$$

2. Type 2: By (4.6) and (4.8), we have $a_n = (-1)^n \delta_{n-1} q(n)$, where

$$q(n) = c_0 n^{s/m}[1 + h(n)] + 1, \quad h(n) \in \tilde{\mathbf{A}}_0^{(-r/m,m)} \text{ strictly.} \tag{4.14}$$

Then we have the following:

$$q(n) = (c_0 + 1) + c_0 h(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \text{ strictly, when } s = 0. \tag{4.15}$$

$$q(n) \in \tilde{\mathbf{A}}_0^{(s/m,m)} \quad \text{strictly, when } s > 0. \tag{4.16}$$

$$q(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \quad \text{strictly, when } s < 0. \tag{4.17}$$

Finally, let us write a_n in the form

$$a_n = \delta_n t(n) \quad \text{for type 1;} \quad a_n = (-1)^n \delta_n t(n) \quad \text{for type 2,} \tag{4.18}$$

where

$$t(n) = \frac{q(n)}{\delta_n/\delta_{n-1}} = n^{-s/m} e^{-f(n)} q(n), \tag{4.19}$$

with the appropriate $q(n)$. Invoking (4.9)–(4.17) in (4.18)–(4.19), we conclude that

$$\text{for type 1:} \quad a_n = (n!)^{s/m} e^{Q(n)} n^{\gamma} w(n), \quad w(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \quad \text{strictly,} \tag{4.20}$$

$$\begin{aligned} \gamma = -r/m \quad \text{if } s = 0 \text{ and } \theta_0 = 0; \quad \gamma = 0 \quad \text{if } s = 0 \text{ and } \theta_0 \neq 0;\\ \gamma = 0 \quad \text{if } s > 0; \quad \gamma = |s|/m \quad \text{if } s < 0. \end{aligned} \tag{4.21}$$

$$\text{for type 2:} \quad a_n = (n!)^{s/m} e^{\widetilde{Q}(n)} n^{\gamma} w(n), \quad w(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \quad \text{strictly,} \tag{4.22}$$

$$\gamma = 0 \quad \text{if } s \geq 0; \quad \gamma = |s|/m \quad \text{if } s < 0, \tag{4.23}$$

$$\widetilde{Q}(n) = \mathrm{i}\pi n + Q(n) = (\mathrm{i}\pi + \theta_0)n + \sum_{i=r}^{m-1} \theta_i n^{1-i/m}. \tag{4.24}$$

Here, we have made use of the fact that $(-1)^n = e^{\mathrm{i}\pi n}$.

This completes the asymptotic analysis of a_n in all the different situations. Clearly, $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$ for all the cases studied.

4.2 Analysis of $A_n = \sum_{k=1}^n a_k$

We now turn to the asymptotic analysis of $A_{n-1} = \sum_{k=1}^{n-1} a_k$. Let us first express (4.1) and (4.2), respectively, as in

$$A_{n-1} = -\delta_0 + \delta_{n-1} \quad \text{for type 1} \tag{4.25}$$

and

$$A_{n-1} = -\delta_0 + (-1)^{n-1}\delta_{n-1} \quad \text{for type 2.} \tag{4.26}$$

Invoking now (i) $\delta_{n-1} = a_n/q(n)$ for type 1 and (ii) $\delta_{n-1} = (-1)^n a_n/q(n)$ for type 2, from (4.5) and (4.6), respectively, and invoking also (4.10)–(4.17), we obtain for both type 1 and type 2 series

$$A_{n-1} = -\delta_0 + n^{\sigma} a_n g(n), \quad g(n) \in \tilde{\mathbf{A}}_0^{(0,m)} \quad \text{strictly,} \tag{4.27}$$

with σ assuming the following values:

For type 1:

$$\sigma = r/m \quad \text{if } s = 0 \text{ and } \theta_0 = 0; \qquad \sigma = 0 \quad \text{if } s = 0 \text{ and } \theta_0 \neq 0;$$
$$\sigma = -s/m \quad \text{if } s > 0; \qquad \sigma = 0 \quad \text{if } s < 0. \tag{4.28}$$

For type 2:

$$\sigma = 0 \quad \text{if } s = 0; \quad \sigma = -s/m \quad \text{if } s > 0; \quad \sigma = 0 \quad \text{if } s < 0. \tag{4.29}$$

These are clearly consistent with Theorem 2.6 when $\delta_n = (n!)^{s/m} e^{Q(n)}$ with either $s < 0$ or $s = 0$ and $\lim_{n\to\infty} \Re Q(n) = -\infty$; in such cases, the series $\sum_{n=1}^{\infty} a_n$ converge. Of course, Theorem 2.6 does not directly apply to the (strongly) divergent cases for which $s > 0$ or $s = 0$ and $\lim_{n\to\infty} \Re Q(n) = +\infty$, but seems to cover them too. It does so in the cases described in (4.1)–(4.3) we have just studied.

We would like to note that, in all the cases considered above, $-\delta_0$ is the sum of the infinite series $\sum_{n=1}^{\infty} a_n$ when this series converges, that is, when $\lim_{n\to\infty} A_n$ exists; it seems to be the antilimit of $\{A_n\}$ when $\lim_{n\to\infty} A_n$ does not exist, and numerical experiments confirm this assertion.

5 Numerical examples I

We have applied the $\tilde{d}^{(m)}$ transformation to various infinite series $\sum_{n=1}^{\infty} a_n$ with $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$ for various values of $m \geq 2$ and verified that it is an effective convergence accelerator.

In this section, we report numerical results obtained from fourteen different series with $m = 2$. Specifically, we treat several telescoping series of the types considered in the preceding section, namely, those with $a_n = \delta_n - \delta_{n-1}$ for type 1 series and $a_n = (-1)^n(\delta_n + \delta_{n-1})$ for type 2 series, for which the limits or antilimits are known to be $S = -\delta_0$. We also treat examples of divergent series with unknown antilimits.[13] In our examples, we have both (i) $s = 0$ and $s \neq 0$, (ii) $Q(n) \equiv 0$ and $Q(n) = \theta_0 n \pm \sqrt{n}$, (with both $\theta_0 = 0$ and $\theta_0 \neq 0$), and (iii) $\gamma = 0$ and $\gamma \neq 0$; we observe different numerical stability issues depending on whether $s = 0$ or not, $Q(n) \equiv 0$ or not, and in case $Q(n) \not\equiv 0$, we observe different behavior whether $\theta_0 = 0$ or not. The fact that there are several different cases, each being convergent and divergent, and each having its own convergence and stability characteristics, accounts for the large number of the numerical examples we give in this section. Note that each example illustrates only one of the many different cases discussed in the preceding sections.

As mentioned earlier, we can replace $\hat{\sigma}$ in (3.5) by 1, that is, $\omega_r = ra_r$ in the W-algorithm of Section 3.4, and this is what we have done here.[14] This way we do not have to worry about the exact value of σ in (3.1). We also recall that, with $\hat{\sigma} = 1$ in

[13] For the divergent series considered here, we do not even know whether antilimits exist. The approximations $A_n^{(0)}$, $n = 0, 1, \ldots$, obtained by applying the $\tilde{d}^{(m)}$ transformation to these series seem definitely to converge, however. Thus, we can safely conclude that $\lim_{n\to\infty} A_n^{(0)}$ are the antilimits of these series, even though we do not know their nature.

[14] See footnote 6.

(3.5), we do not need any further information about $Q(n)$ and the parameters s, r, and γ in (1.1); mere knowledge of the fact that $\{a_n\}$ is in $\tilde{\mathbf{b}}^{(m)}$ is sufficient for applying the $\tilde{d}^{(m)}$ transformation.

We have done all our computations using quadruple-precision arithmetic, for which the roundoff unit is $\mathbf{u} = 1.93 \times 10^{-34}$. This means that the highest number of significant decimal digits we can have is about 34. In addition, if $|\Lambda_n^{(j)}/\bar{A}_n^{(j)}| = O(10^q)$ for some $q > 0$, then the number of correct significant figures in $\bar{A}_n^{(j)}$ is about $p = 34 - q$, close to convergence. The tables for our numerical examples amply demonstrate the correctness of this conclusion. We advise the reader to pay attention to this fact.

In all the examples, we first try the $\tilde{d}^{(m)}$ transformation with $R_l = l + 1$, $l = 0, 1, \ldots,$ which is the simplest and most immediate choice for the R_l. As we will see, there are some slow convergence and numerical stability issues with some of these examples when the R_l are chosen this way. We demonstrate that these two issues can be treated simultaneously in an effective way by using APS in some cases and GPS in some others. In addition, it will become obvious from our numerical results that the relative error assessments shown in (3.12) (with $\Gamma_n^{(j)}$) and in (3.18) (with $\Lambda_n^{(j)}$) are very reliable. This clearly demonstrates the relevance and importance of the $\Gamma_n^{(j)}$ and $\Lambda_n^{(j)}$ in assessing the accuracy of the numerical approximations to limits or antilimits. Again, the fact that the $\Gamma_n^{(j)}$ and $\Lambda_n^{(j)}$ can be obtained recursively via the W-algorithm, and simultaneously with the approximations $A_n^{(j)}$, is really surprising.

We have applied APS with

$$\text{(APS):}\quad R_l = \kappa(l+1), \quad l = 0, 1, \ldots; \quad \text{integer } \kappa \geq 1.$$

We have applied GPS with

$$\text{(GPS):}\quad R_0 = 1; \quad R_l = \max\{\lfloor \tau R_{l-1} \rfloor, l+1\}, \quad l = 1, 2, \ldots; \quad \tau \in (1, 2).$$

As usual, $A_n = \sum_{k=1}^{n} a_k$ and $A_n^{(j)} \equiv \tilde{d}_n^{(m,j)}$, in the tables that accompany the examples. Recall also that R_{j+n} is the number of the terms of the series used for constructing $\tilde{d}_n^{(m,j)}$.

As a rule of thumb, we can reach the following conclusions:

C1. If $a_n = h(n) \in \tilde{\mathbf{A}}_0^{(\gamma,m)}$, use GPS.

C2. If $a_n = \exp[Q(n)]n^\gamma w(n)$, where $Q(n) = \sum_{i=1}^{m-1} \theta_i n^{1-i/m}$ (i.e. $\theta_0 = 0$), and $w(n) \in \tilde{\mathbf{A}}_0^{(0,m)}$, use GPS.

C3. If $a_n = \exp[Q(n)]n^\gamma w(n)$, where $Q(n) = \sum_{i=0}^{m-1} \theta_i n^{1-i/m}$ (with $\theta_0 \neq 0$), and $w(n) \in \tilde{\mathbf{A}}_0^{(0,m)}$, use APS.

C4. If $a_n = (n!)^{s/m} \exp[Q(n)]n^\gamma w(n)$, where $s \neq 0$, $Q(n) = \sum_{i=0}^{m-1} \theta_i n^{1-i/m}$, and $w(n) \in \tilde{\mathbf{A}}_0^{(0,m)}$, use GPS.

C5. If a_n is as in any of the cases C1–C4 (with real θ_0) multiplied by $(-1)^n$ for all n, use APS with $R_l = l + 1$, $l = 0, 1, \ldots$. Note that, in these cases, $\Gamma_n^{(j)} = 1$ in accordance with (3.13).

Table 2 Numerical results for Example 5.1 [$a_n = e^{-\sqrt{n}} - e^{-\sqrt{n-1}}$], where the R_l are chosen using APS as $R_l = l + 1, l = 0, 1, \dots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	3.68D−01	3.68D−01	1.00D+00	6.32D−01
4	5	1.07D−01	7.64D−02	2.90D+02	2.47D+02
8	9	4.98D−02	4.65D−04	3.03D+04	2.80D+04
12	13	2.72D−02	1.17D−06	4.58D+06	4.37D+06
16	17	1.62D−02	1.48D−09	7.95D+08	7.72D+08
20	21	1.02D−02	1.13D−12	1.53D+11	1.50D+11
24	25	6.74D−03	5.82D−16	3.19D+13	3.14D+13
28	29	4.58D−03	4.03D−19	7.10D+15	7.03D+15
32	33	3.20D−03	2.40D−17	1.67D+18	1.66D+18
36	37	2.28D−03	9.81D−15	4.13D+20	4.11D+20
40	41	1.66D−03	1.03D−12	1.07D+23	1.06D+23

Of course, in all cases, we can try $R_l = l + 1$, $l = 0, 1, \dots,$ first. We use the classification C1–C5 in our examples below.

Remark 7 Before ending, we would like to re-emphasize the following points:

1. The only assumption we make when applying the $\tilde{d}^{(m)}$ transformation to $\sum_{n=1}^{\infty} a_n$ is that the sequence $\{a_n\}$ is in $\tilde{\mathbf{b}}^{(m)}$ for some m; no further information about the specific parameters $[s, \gamma, Q(n)]$ in the asymptotic expansion of a_n as $n \to \infty$ is needed or is being used in the computation. We are also using the most user-friendly definition of the $\tilde{d}^{(m)}$ transformation with $\hat{\sigma} = 1$, without having to know the exact σ.

Table 3 Numerical results for Example 5.1 [$a_n = e^{-\sqrt{n}} - e^{-\sqrt{n-1}}$], where the R_l are chosen using GPS with $\tau = 1.3$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	3.68D−01	3.68D−01	1.00D+00	6.32D−01
4	5	1.07D−01	7.64D−02	2.90D+02	2.47D+02
8	11	3.63D−02	3.27D−04	8.41D+03	7.72D+03
12	29	4.58D−03	7.32D−08	4.26D+03	4.10D+03
16	80	1.30D−04	4.43D−13	3.75D+02	3.73D+02
20	227	2.86D−07	3.11D−20	1.80D+01	1.79D+01
24	646	9.16D−12	6.18D−30	2.25D+00	2.25D+00
28	1842	2.29D−19	0.00D+00	1.10D+00	1.10D+00
32	5258	3.43D−32	1.64D−33	1.00D+00	1.00D+00

Table 4 Numerical results for Example 5.2 [$a_n = (-1)^n(e^{-\sqrt{n}}+e^{-\sqrt{n-1}})$], using $R_l = l+1, l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert \bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	3.68D−01	3.68D−01	1.00D+00	1.37D+00
4	5	1.07D−01	4.16D−04	1.00D+00	1.00D+00
8	9	4.98D−02	3.33D−08	1.00D+00	1.00D+00
12	13	2.72D−02	6.71D−13	1.00D+00	1.00D+00
16	17	1.62D−02	5.61D−18	1.00D+00	1.00D+00
20	21	1.02D−02	2.48D−23	1.00D+00	1.00D+00
24	25	6.74D−03	6.72D−29	1.00D+00	1.00D+00
28	29	4.58D−03	4.81D−34	1.00D+00	1.00D+00
32	33	3.20D−03	3.85D−34	1.00D+00	1.00D+00

2. The input needed for computing $\Gamma_n^{(j)}$ and $\Lambda_n^{(j)}$ is precisely that used to compute $A_n^{(j)}$; namely, the terms $a_1, \ldots, a_{R_{j+n}}$. Nothing else is needed. Thus, all three quantities can be computed simultaneously and efficiently by the recursive W-algorithm.
3. We also recall that when $|\Lambda_n^{(j)}/\bar{A}_n^{(j)}|\mathbf{u} = O(10^{-p})$ for some positive integer p, we can conclude safely that, as an approximation to S, $\bar{A}_n^{(j)}$ has approximately p correct significant figures, close to convergence. The numbers in the tables obtained from all of the examples below show this to be the case both (i) when the series $\sum_{n=1}^{\infty} a_n$ converge and also (ii) when they diverge, weakly or strongly. To illustrate this important point, let us look at two of the (C2) examples below:

Table 5 Numerical results for Example 5.3 [$a_n = e^{\sqrt{n}} - e^{\sqrt{n-1}}$], using $R_l = l + 1, l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert \bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.72D+00	2.72D+00	1.00D+00	1.72D+00
4	5	9.36D+00	2.60D+00	3.13D+02	1.65D+03
8	9	2.01D+01	2.10D+00	1.03D+06	1.17D+07
12	13	3.68D+01	5.08D−01	5.04D+09	1.01D+11
16	17	6.18D+01	8.73D−03	4.72D+12	1.51D+14
20	21	9.78D+01	1.64D−04	9.85D+15	4.75D+17
24	25	1.48D+02	3.52D−06	4.11D+19	2.86D+21
28	29	2.18D+02	4.14D−09	1.23D+22	1.20D+24
32	33	3.12D+02	1.67D−06	2.04D+25	2.73D+27
36	37	4.38D+02	2.14D−03	8.15D+28	1.46D+31
40	41	6.04D+02	1.52D+01	2.76D+31	6.56D+33

Table 6 Numerical results for Example 5.3 [$a_n = e^{\sqrt{n}} - e^{\sqrt{n-1}}$], where the R_l are chosen using GPS with $\tau = 1.3$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.72D+00	2.72D+00	1.00D+00	1.72D+00
4	5	9.36D+00	2.60D+00	3.13D+02	1.65D+03
8	11	2.76D+01	2.09D+00	4.93D+05	5.04D+06
12	29	2.18D+02	3.33D−01	4.81D+07	8.42D+08
16	80	7.66D+03	8.33D−04	7.10D+07	4.11D+09
20	227	3.49D+06	9.80D−08	1.18D+07	4.57D+09
24	646	1.09D+11	1.16D−11	5.10D+06	3.60D+10
28	1842	4.36D+18	3.04D−16	1.23D+06	5.38D+11
32	5258	3.10D+31	2.83D−19	2.73D+05	2.55D+13

- In Example 5.3, for which the antilimit of the divergent series $\sum_{n=1}^{\infty}(e^{\sqrt{n}} - e^{\sqrt{n-1}})$ seems to be $S = -1$, we have the following: In Table 5, $|\Lambda_{28}^{(0)}/\bar{A}_{28}^{(0)}|\mathbf{u} \approx O(10^{24-34}) = O(10^{-10})$, consistent with $|\bar{A}_{28}^{(0)} - S|/|S| = O(10^{-9})$. In Table 6, $|\Lambda_{32}^{(0)}/\bar{A}_{32}^{(0)}|\mathbf{u} \approx O(10^{13-34}) = O(10^{-21})$, consistent with $|\bar{A}_{32}^{(0)} - S|/|S| = O(10^{-19})$.
- In Example 5.5, for which the antilimit of the divergent series $\sum_{n=1}^{\infty} e^{\sqrt{n}}$ seems to be $S = 1.24628299466148185\cdots$, up to 18 decimal digits, we have the following: In Table 8, $|\Lambda_{28}^{(0)}/\bar{A}_{28}^{(0)}|\mathbf{u} \approx O(10^{25-34}) = O(10^{-9})$, consistent with $\bar{A}_{28}^{(0)} = 1.24628299\cdots$ (the first 9 digits of S). In Table 9, $|\Lambda_{32}^{(0)}/\bar{A}_{32}^{(0)}|\mathbf{u} \approx O(10^{15-34}) = O(10^{-19})$, consistent with $\bar{A}_{28}^{(0)} = 1.24628299466148185\cdots$ (the first 18 digits of S).

Table 7 Numerical results for Example 5.4 [$a_n = (-1)^n(e^{\sqrt{n}} + e^{\sqrt{n-1}})$], using $R_l = l + 1, l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.72D+00	2.72D+00	1.00D+00	3.72D+00
4	5	9.36D+00	1.17D−02	1.00D+00	5.94D+00
8	9	2.01D+01	4.17D−06	1.00D+00	1.20D+01
12	13	3.68D+01	2.55D−10	1.00D+00	2.06D+01
16	17	6.18D+01	5.40D−15	1.00D+00	3.24D+01
20	21	9.78D+01	5.43D−20	1.00D+00	4.85D+01
24	25	1.48D+02	3.07D−25	1.00D+00	6.98D+01
28	29	2.18D+02	1.10D−30	1.00D+00	9.76D+01
32	33	3.12D+02	3.85D−33	1.00D+00	1.33D+02

Table 8 Numerical results for Example 5.5 [$a_n = e^{\sqrt{n}}$], using $R_l = l + 1, l = 0, 1, \ldots$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.72D+00	2.71828182845904523536028747135266231D+00	1.00D+00	2.72D+00
4	5	2.92D+01	1.25526985654591147597524366280725429D+00	1.36D+03	1.95D+04
8	9	9.19D+01	1.24677273910725248869144831373621801D+00	3.82D+06	1.69D+08
12	13	2.12D+02	1.24627554920630442266529384719068750D+00	1.07D+10	1.03D+12
16	17	4.18D+02	1.24628333168062508590422433878661035D+00	5.25D+13	9.49D+15
20	21	7.52D+02	1.24628299284466085557931505666022122D+00	1.41D+16	4.35D+18
24	25	1.26D+03	1.24628299466999557367176182613152417D+00	1.73D+19	8.56D+21
28	29	2.03D+03	1.24628299198356711403961152832133262D+00	9.02D+22	6.81D+25
32	33	3.12D+03	1.24628324241958617654595744535696746D+00	4.54D+25	5.06D+28

Example 5.1 Let $a_n = e^{-\sqrt{n}} - e^{-\sqrt{n-1}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C2 category and *converges* with limit $S = -1$. Table 2 contains results obtained by choosing $R_l = l + 1, l = 0, 1, \ldots$. In Table 3, we present results obtained by choosing the R_l using GPS with $\tau = 1.3$.

Example 5.2 Let $a_n = (-1)^n(e^{-\sqrt{n}} + e^{-\sqrt{n-1}})$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C2/C5 category and *converges* with limit $S = -1$. Table 4 contains results obtained by choosing $R_l = l + 1, l = 0, 1, \ldots$.

Example 5.3 Let $a_n = e^{\sqrt{n}} - e^{\sqrt{n-1}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C2 category and *diverges* with apparent antilimit $S = -1$. Table 5 contains results obtained by choosing $R_l = l+1, l = 0, 1, \ldots$. In Table 6, we present results obtained by choosing the R_l using GPS with $\tau = 1.3$.

Table 9 Numerical results for Example 5.5 [$a_n = e^{\sqrt{n}}$], where the R_l are chosen using GPS with $\tau = 1.3$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.72D+00	2.71828182845904523536028747135266231D+00	1.00D+00	2.72D+00
4	5	2.92D+01	1.25526985654591147597524366280725429D+00	1.36D+03	1.95D+04
8	11	1.43D+02	1.24671505638378127944673235454440382D+00	1.57D+06	5.94D+07
12	29	2.03D+03	1.24628218609008659057856308239992091D+00	2.73D+07	2.12D+09
16	80	1.26D+05	1.24628298810418632879233471192843094D+00	3.94D+07	1.36D+10
20	227	1.00D+08	1.24628299465099003773467908406255098D+00	1.53D+08	5.13D+11
24	646	5.39D+12	1.24628299466148082969324108233096056D+00	7.47D+06	6.44D+11
28	1842	3.68D+20	1.24628299466148185365143315380862845D+00	1.79D+06	1.31D+13
32	5258	4.45D+33	1.24628299466148185195182683756923951D+00	1.84D+06	3.78D+15

Table 10 Numerical results for Example 5.6 [$a_n = (-1)^n e^{\sqrt{n}}$], using $R_l = l + 1, l = 0, 1, \ldots$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	−2.72D+00	−2.71828182845904523536028747135266231D+00	1.00D+00	2.72D+00
4	5	−6.22D+00	−1.02389938149412728674621319253264616D+00	1.00D+00	3.42D+00
8	9	−1.19D+01	−1.02396073254025925488406911517938308D+00	1.00D+00	6.63D+00
12	13	−2.07D+01	−1.02396073204910424017949474942021129D+00	1.00D+00	1.12D+01
16	17	−3.38D+01	−1.02396073204906060742047437248544015D+00	1.00D+00	1.74D+01
20	21	−5.26D+01	−1.02396073204906060526543006429244242D+00	1.00D+00	2.58D+01
24	25	−7.89D+01	−1.02396073204906060526534757271439465D+00	1.00D+00	3.70D+01
28	29	−1.15D+02	−1.02396073204906060526534757003586791D+00	1.00D+00	5.15D+01
32	33	−1.64D+02	−1.02396073204906060526534757003580917D+00	1.00D+00	7.01D+01

Example 5.4 Let $a_n = (-1)^n(e^{\sqrt{n}} + e^{\sqrt{n-1}})$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C2/C5 category and *diverges* with apparent antilimit $S = -1$. Table 7 contains results obtained by choosing $R_l = l + 1, l = 0, 1, \ldots$.

Example 5.5 Let $a_n = e^{\sqrt{n}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C2 category and *diverges*, possibly with an antilimit S that is not known. Table 8 contains results obtained by choosing $R_l = l+1, l = 0, 1, \ldots$. In Table 9, we present results obtained by choosing the R_l using GPS with $\tau = 1.3$.

Example 5.6 Let $a_n = (-1)^n e^{\sqrt{n}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C2/C5 category and *diverges*, possibly with an antilimit S that is not known. Table 10 contains results obtained by choosing $R_l = l + 1, l = 0, 1, \ldots$.

Table 11 Numerical results for Example 5.7 [$a_n = e^{-0.2n+\sqrt{n}} - e^{-0.2(n-1)+\sqrt{n-1}}$], using $R_l = l + 1$, $l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.23D+00	2.23D+00	1.00D+00	1.23D+00
8	9	3.32D+00	3.48D+00	1.31D+00	3.24D+00
16	17	2.06D+00	3.48D+00	9.01D+01	1.69D+02
24	25	1.00D+00	3.48D+00	1.21D+07	1.02D+07
32	33	4.25D−01	3.45D+00	4.43D+13	8.27D+12
40	41	1.66D−01	2.46D−02	8.99D+18	3.96D+18
48	49	6.08D−02	3.83D−07	1.74D+22	1.25D+22
56	57	2.13D−02	5.44D−10	3.32D+25	2.87D+25
64	65	7.17D−03	9.33D−06	6.27D+28	5.87D+28

Table 12 Numerical results for Example 5.7 [$a_n = e^{-0.2n+\sqrt{n}} - e^{-0.2(n-1)+\sqrt{n-1}}$], where the R_l are chosen using APS with $\kappa = \eta = 5$; that is, $R_l = 5(l + 1)$, $l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\|A_{R_n} - S\|/\|S\|$	$\|\bar{A}_n^{(0)} - S\|/\|S\|$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	5	3.44D+00	3.44D+00	1.00D+00	2.44D+00
4	25	1.00D+00	4.08D+00	1.47D+01	1.54D+01
8	45	1.01D−01	1.47D−02	2.50D+02	1.59D+02
12	65	7.17D−03	1.01D−06	1.03D+03	9.79D+02
16	85	4.18D−04	1.91D−11	4.53D+03	4.50D+03
20	105	2.14D−05	1.48D−16	2.00D+04	2.00D+04
24	125	9.96D−07	5.84D−22	8.86D+04	8.86D+04
28	145	4.32D−08	1.34D−27	3.90D+05	3.90D+05
32	165	1.77D−09	5.29D−29	1.71D+06	1.71D+06

Example 5.7 Let $a_n = e^{-0.2n+\sqrt{n}} - e^{-0.2(n-1)+\sqrt{n-1}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C3 category and *converges* with limit $S = -1$. Table 11 contains results obtained by choosing $R_l = l + 1$, $l = 0, 1, \ldots$. In Table 12 we present results obtained by choosing the R_l using APS with $\kappa = \eta = 5$, thus $R_l = 5(l + 1)$, $l = 0, 1, \ldots$.

Example 5.8 Let $a_n = (-1)^n(e^{-0.2n+\sqrt{n}} + e^{-0.2(n-1)+\sqrt{n-1}})$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C3/C5 category and *converges* with limit $S = -1$. Table 13 contains results obtained by choosing $R_l = l + 1$, $l = 0, 1, \ldots$.

Example 5.9 Let $a_n = e^{-0.2n+\sqrt{n}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C3 category and *converges* to a limit S that is not known. Table 14 contains results obtained

Table 13 Numerical results for Example 5.8 [$a_n = (-1)^n(e^{-0.2n+\sqrt{n}} + e^{-0.2(n-1)+\sqrt{n-1}})$], using $R_l = l + 1$, $l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\|A_{R_n} - S\|/\|S\|$	$\|\bar{A}_n^{(0)} - S\|/\|S\|$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.23D+00	2.23D+00	1.00D+00	3.23D+00
4	5	3.44D+00	7.30D−03	1.00D+00	3.12D+00
8	9	3.32D+00	1.43D−06	1.00D+00	3.45D+00
12	13	2.73D+00	4.81D−11	1.00D+00	3.24D+00
16	17	2.06D+00	5.60D−16	1.00D+00	2.80D+00
20	21	1.47D+00	3.09D−21	1.00D+00	2.30D+00
24	25	1.00D+00	9.60D−27	1.00D+00	1.82D+00
28	29	6.60D−01	2.06D−32	1.00D+00	1.40D+00
32	33	4.25D−01	2.12D−33	1.00D+00	1.12D+00

Table 14 Numerical results for Example 5.9 [$a_n = e^{-0.2n+\sqrt{n}}$], using $R_l = l + 1, l = 0, 1, \ldots$. Note that the limit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.23D+00	2.22554092849246760457953753139507683D+00	1.00D+00	2.23D+00
8	9	2.85D+01	6.91041025116709498367466955167170379D+01	3.07D+05	6.07D+06
16	17	4.97D+01	6.94975935915328024186339724785494270D+01	6.45D+08	2.51D+10
24	25	6.11D+01	6.94975997601623491994988589397430127D+01	1.42D+12	7.43D+13
32	33	6.62D+01	6.94975997602064996307484910836876701D+01	3.03D+15	1.83D+17
40	41	6.83D+01	6.94975997602064916366549830540673670D+01	6.29D+18	4.09D+20
48	49	6.91D+01	6.94975997601806743910266253659572031D+01	1.27D+22	8.57D+23
56	57	6.94D+01	6.94975997308245777099536081393349112D+01	2.52D+25	1.72D+27
64	65	6.95D+01	6.94975821438721079747682396283208104D+01	4.89D+28	3.37D+30

by choosing $R_l = l + 1, l = 0, 1, \ldots$. In Table 15, we present results obtained by choosing the R_l using APS with $\kappa = \eta = 5$, thus $R_l = 5(l + 1), l = 0, 1, \ldots$.

Example 5.10 Let $a_n = (-1)^n e^{0.2n-\sqrt{n}}, n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C3/C5 category and *diverges*, possibly with an antilimit S that is not known. Table 16 contains results obtained by choosing $R_l = l + 1, l = 0, 1, \ldots$.

Example 5.11 Let $a_n = \sqrt{n!}e^{-\sqrt{n}} - \sqrt{(n-1)!}e^{-\sqrt{n-1}}, n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C4 category and *diverges* with apparent antilimit $S = -1$. Table 17 contains results obtained by choosing $R_l = l + 1, l = 0, 1, \ldots$. In Table 18, we present results obtained by choosing the R_l using GPS with $\tau = 1.1$.

Table 15 Numerical results for Example 5.9 [$a_n = e^{-0.2n+\sqrt{n}}$], where the R_l are chosen using APS with $\kappa = \eta = 5$; that is, $R_l = 5(l + 1), l = 0, 1, \ldots$. Note that the limit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	5	1.48D+01	1.48469162378884426206684133887622417D+01	1.00D+00	1.48D+01
4	25	6.11D+01	6.96296394954322519778004220754137383D+01	5.01D+01	2.57D+03
8	45	6.88D+01	6.94976035518697374035719835887475438D+01	2.02D+02	1.34D+04
12	65	6.95D+01	6.94975997601454883646767731022889718D+01	9.03D+02	6.25D+04
16	85	6.95D+01	6.94975997602064989300261396945842564D+01	4.09D+03	2.84D+05
20	105	6.95D+01	6.94975997602064988540120840350043551D+01	1.84D+04	1.28D+06
24	125	6.95D+01	6.94975997602064988540031066443388019D+01	8.24D+04	5.72D+06
28	145	6.95D+01	6.94975997602064988540031067912671900D+01	3.66D+05	2.54D+07
32	165	6.95D+01	6.94975997602064988540031067892774609D+01	1.62D+06	1.12D+08

Table 16 Numerical results for Example 5.10 [$a_n = (-1)^n e^{0.2n-\sqrt{n}}$], using $R_l = l+1, l = 0, 1, \ldots$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	−4.49D−01	−4.49328964117221591430102385015562784D−01	1.00D+00	4.49D−01
4	5	−3.98D−01	−2.54755240466624695829767367307432419D−01	1.00D+00	2.55D−01
8	9	−4.08D−01	−2.54747573868605037684471045364090490D−01	1.00D+00	2.55D−01
12	13	−4.43D−01	−2.54747573734873382943870173850446560D−01	1.00D+00	2.55D−01
16	17	−5.07D−01	−2.54747573734869455944444615389169251D−01	1.00D+00	2.55D−01
20	21	−6.11D−01	−2.54747573734869455818166677569691476D−01	1.00D+00	2.58D−01
24	25	−7.80D−01	−2.54747573734869455818162487122226260D−01	1.00D+00	2.88D−01
28	29	−1.05D+00	−2.54747573734869455818162486982736643D−01	1.00D+00	3.62D−01
32	33	−1.50D+00	−2.54747573734869455818162486982731683D−01	1.00D+00	4.78D−01
36	37	−2.23D+00	−2.54747573734869455818162486982731443D−01	1.00D+00	6.44D−01
40	41	−3.45D+00	−2.54747573734869455818162486982731587D−01	1.00D+00	8.81D−01

Example 5.12 Let $a_n = (-1)^n(\sqrt{n!}e^{-\sqrt{n}} + \sqrt{(n-1)!}e^{-\sqrt{n-1}})$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C4/C5 category and *diverges* with apparent antilimit $S = -1$. Table 19 contains results obtained by choosing $R_l = l+1, l = 0, 1, \ldots$.

Example 5.13 Let $a_n = (-1)^n\sqrt{n!}e^{-\sqrt{n}}$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C4/C5 category and *diverges*, possibly with an antilimit S that is not known. Table 20 contains results obtained by choosing $R_l = l+1, l = 0, 1, \ldots$.

Example 5.14 Let $a_n = n^{\sqrt{3}}/(1+\sqrt{n})$, $n = 1, 2, \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is in the C1 category and *diverges* with an antilimit S that is not known. Table 21 contains results obtained by choosing $R_l = l+1, l = 0, 1, \ldots$. In Table 22 , we present results obtained by choosing the R_l using GPS with $\tau = 1.3$. Note that $a_n = u(n) \in \tilde{\mathbf{A}}_0^{(\sqrt{3}-1/2,2)}$ and satisfies Theorem 2.2 with $c = 0$ and b the antilimit in (2.2), thus $\{a_n\} \in \tilde{\mathbf{b}}^{(2)}$ by part 1 of Theorem 2.5.

6 Application to computation of infinite products

The machinery of the class $\tilde{\mathbf{b}}^{(m)}$ and the $\tilde{d}^{(m)}$ transformation treated above can also be used to accelerate the convergence of some infinite products, as discussed briefly in [18, Section 25.11]. Here, we expand on the treatment of [18] considerably. We deal with convergent infinite products[15] of the form

[15]Recall that the infinite product $\prod_{n=1}^{\infty} f_n$ is convergent if $\lim_{n\to\infty}\prod_{k=1}^{n} f_k$ exists and is finite and nonzero.

Table 17 Numerical results for Example 5.11 [$a_n = \sqrt{n!}e^{-\sqrt{n}} - \sqrt{(n-1)!}e^{-\sqrt{n-1}}$], using $R_l = l+1$, $l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	3.68D−01	3.68D−01	1.00D+00	6.32D−01
4	5	1.17D+00	3.51D−01	1.36D+00	7.80D−01
8	9	3.00D+01	3.49D−01	2.69D+01	3.27D+01
12	13	2.14D+03	3.19D−01	4.62D+03	7.39D+04
16	17	3.05D+05	7.75D−01	5.67D+06	1.17D+09
20	21	7.31D+07	7.03D−04	4.73D+06	1.58D+10
24	25	2.65D+10	4.63D−06	4.41D+07	2.91D+12
28	29	1.36D+13	3.89D−07	7.38D+09	1.14D+16
32	33	9.43D+15	9.10D−11	4.51D+09	1.89D+17
36	37	8.47D+18	4.21D−13	6.94D+10	8.89D+19
40	41	9.58D+21	1.53D−12	4.90D+11	2.14D+22

$$S = \prod_{n=1}^{\infty}(1+v_n), \quad v_n = w(n) \in \tilde{\mathbf{A}}_0^{(-t/m,m)} \text{ strictly}, \quad t \geq m+1 \text{ integer.} \tag{6.1}$$

Recall that the infinite product converges if and only if $\sum_{k=1}^{\infty} v_k$ converges, which implies that $t/m > 1$, which in turn implies that $t \geq m+1$ since t is an integer.

Let us define

$$A_0 = 0; \quad A_n = \prod_{k=1}^{n}(1+v_k), \quad n = 1, 2, \ldots; \quad a_n \equiv A_n - A_{n-1}, \quad n = 1, 2, \ldots. \tag{6.2}$$

Then,

$$A_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, \ldots, \quad \text{and} \quad S = \lim_{n\to\infty} A_n. \tag{6.3}$$

Now,

$$A_n = (1+v_n)A_{n-1}, \quad n = 2, 3, \ldots.$$

Therefore,

$$a_n \equiv A_n - A_{n-1} = v_n A_{n-1} \quad \Rightarrow \quad A_{n-1} = \frac{a_n}{v_n}, \quad n = 2, 3, \ldots. \tag{6.4}$$

Applying Δ to both sides of (6.4), we obtain

$$a_n = \Delta A_{n-1} = \Delta(a_n/v_n) \quad \Rightarrow \quad a_n = a_{n+1}/v_{n+1} - a_n/v_n, \tag{6.5}$$

which can be written as in

$$a_n = p(n)\Delta a_n; \quad p(n) = [v_{n+1} + (\Delta v_n)/v_n]^{-1}. \tag{6.6}$$

Now, since $v_n \in \tilde{\mathbf{A}}_0^{(-t/m,m)}$ strictly by (6.1), we also have $(\Delta v_n)/v_n \in \tilde{\mathbf{A}}_0^{(-1,m)}$ strictly. In addition, $-t/m < -1$. Therefore, $1/p(n) \in \tilde{\mathbf{A}}_0^{(-1,m)}$ strictly, implying

Table 18 Numerical results for Example 5.11 [$a_n = \sqrt{n!}e^{-\sqrt{n}} - \sqrt{(n-1)!}e^{-\sqrt{n-1}}$], where the R_l are chosen using GPS with $\tau = 1.1$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert \bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	3.68D−01	3.68D−01	1.00D+00	6.32D−01
4	5	1.17D+00	3.51D−01	1.36D+00	7.80D−01
8	9	3.00D+01	3.49D−01	2.69D+01	3.27D+01
12	13	2.14D+03	3.19D−01	4.62D+03	7.39D+04
16	17	3.05D+05	7.75D−01	5.67D+06	1.17D+09
20	22	3.08D+08	6.83D−04	4.35D+06	1.34D+10
24	30	6.81D+13	6.10D−06	2.88D+07	7.03D+11
28	42	5.74D+22	2.52D−08	6.18D+07	6.43D+12
32	60	3.95D+37	1.93D−10	1.72D+08	5.79D+13
36	86	1.46D+61	1.02D−12	2.38D+08	3.10D+14
40	124	5.66D+98	3.00D−15	1.44D+08	1.42D+15
44	179	5.17D+157	9.37D−18	7.85D+07	1.27D+16
48	259	1.24D+250	2.65D−17	6.87D+07	4.18D+17

that $p(n) \in \tilde{\mathbf{A}}_0^{(1,m)}$ strictly. This means that $\{a_n\} \in \tilde{\mathbf{b}}^{(m)}$ by Definition 2.3. Consequently, the $\tilde{d}^{(m)}$ transformation can be applied to the sequence $\{A_n\}$, hence to the series $\sum_{n=1}^{\infty} a_n$, successfully.

Let us now investigate the asymptotic nature of a_n in more detail. We will be applying Theorem 2.4 for this purpose. From (6.5), we have, in addition to (6.6),

$$a_{n+1} = c(n)a_n; \quad c(n) = (1 + 1/v_n)v_{n+1}, \tag{6.7}$$

which, making use of the fact that $v_{n+1} = v_n + \Delta v_n$, we can also write as

$$c(n) = 1 + (\Delta v_n)/v_n + v_n + \Delta v_n. \tag{6.8}$$

Table 19 Numerical results for Example 5.12 [$a_n = (-1)^n(\sqrt{n!}e^{-\sqrt{n}} + \sqrt{(n-1)!}e^{-\sqrt{n-1}})$], using $R_l = l + 1, l = 0, 1, \ldots$. Note that $S = -1$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert \bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	3.68D−01	3.68D−01	1.00D+00	1.37D+00
4	5	1.17D+00	9.49D−04	1.00D+00	9.99D−01
8	9	3.00D+01	6.69D−07	1.00D+00	2.50D+00
12	13	2.14D+03	1.81D−10	1.00D+00	1.96D+01
16	17	3.05D+05	2.79D−14	1.00D+00	2.37D+02
20	21	7.31D+07	2.87D−18	1.00D+00	3.76D+03
24	25	2.65D+10	2.16D−22	1.00D+00	7.38D+04
28	29	1.36D+13	1.25D−26	1.00D+00	1.72D+06
32	33	9.43D+15	1.95D−28	1.00D+00	4.64D+07

Table 20 Numerical results for Example 5.13 [$a_n = (-1)^n\sqrt{n!}e^{-\sqrt{n}}$], using $R_l = l + 1$, $l = 0, 1, \ldots$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	−3.68D−01	−3.67879441171442321595523770161460873D−01	1.00D+00	3.68D−01
4	5	−9.65D−01	−2.05445994186455599969353796265963566D−01	1.00D+00	3.18D−01
8	9	−2.18D+01	−2.05408671703611238491707363340804563D−01	1.00D+00	1.55D+00
12	13	−1.63D+03	−2.05408680194850300292719482233526514D−01	1.00D+00	1.37D+01
16	17	−2.40D+05	−2.05408680199979555770776037771120370D−01	1.00D+00	1.71D+02
20	21	−5.88D+07	−2.05408680199983779970682276964493250D−01	1.00D+00	2.80D+03
24	25	−2.17D+10	−2.05408680199983784675881316821330617D−01	1.00D+00	5.62D+04
28	29	−1.13D+13	−2.05408680199983784682423139886832961D−01	1.00D+00	1.33D+06
32	33	−7.93D+15	−2.05408680199983784682433513602872466D−01	1.00D+00	3.64D+07

Again, since $v_n \in \tilde{\mathbf{A}}_0^{(-t/m,m)}$ strictly by (6.1), we have also $\Delta v_n \in \tilde{\mathbf{A}}_0^{(-1-t/m,m)}$ strictly, as a result of which, we conclude that $(\Delta v_n)/v_n \in \tilde{\mathbf{A}}_0^{(-1,m)}$ strictly. In addition, since $v_n \sim en^{-t/m}$ as $n \to \infty$, for some constant $e \neq 0$, we have $\Delta v_n \sim e(-t/m)n^{-1-t/m}$ as $n \to \infty$, and, therefore, $(\Delta v_n)/v_n \sim (-t/m)n^{-1}$ as $n \to \infty$. Invoking also the fact that $t \geq m + 1$, we finally have that

$$c(n) = 1 - \frac{t}{m}n^{-1} + O(n^{-1-1/m}) \quad \text{as } n \to \infty. \tag{6.9}$$

Thus, Theorem 2.4 holds with $c_0 = 1$ and $c_1 = \cdots = c_{m-1} = 0$ and $c_m = -t/m$. In addition, (2.8) in Theorem 2.4 gives $\epsilon_1 = \cdots = \epsilon_{m-1} = 0$ and $\epsilon_m = -t/m$. Substituting all these into (2.7), we obtain $\mu = 0$, $\theta_0 = \theta_1 = \cdots = \theta_{m-1} = 0$, which implies $Q(n) \equiv 0$, and $\gamma = -t/m$. As a result, (2.4) gives $a_n = h(n)$, $h \in \tilde{\mathbf{A}}_0^{(-t/m,m)}$ strictly.

Table 21 Numerical results for Example 5.14 [$a_n = n^{\sqrt{3}}/(1+\sqrt{n})$], using $R_l = l+1$, $l = 0, 1, \ldots$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	5.00D−01	5.00000000000000000000000000000000000D−01	1.00D+00	5.00D−01
4	5	1.30D+01	1.36953249947897007206016802576582650D−02	1.05D+02	4.99D+02
8	9	4.83D+01	2.14934104105471257196163245799915443D+01	5.87D+09	1.24D+11
12	13	1.11D+02	−6.36685109382204842916350986273167363D+00	1.48D+13	7.52D+14
16	17	2.05D+02	−6.33514738456051770232899035920678815D+00	9.82D+16	9.45D+18
20	21	3.31D+02	−6.33490882742039134712027408229034021D+00	4.83D+20	7.64D+22
24	25	4.93D+02	−6.33489970412111983179650131962681705D+00	1.91D+24	4.55D+26
28	29	6.93D+02	−6.33485606274876151464675840216981192D+00	6.50D+27	2.19D+30
32	33	9.31D+02	−6.23505348202801983168713408136424401D+00	1.98D+31	9.00D+33

Table 22 Numerical results for Example 5.14 [$a_n = n^{\sqrt{3}}/(1+\sqrt{n})$], where the R_l are chosen using GPS with $\tau = 1.3$. Note that the antilimit is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	5.00D−01	5.0000000000000000000000000000000000D−01	1.00D+00	5.00D−01
4	5	1.30D+01	1.3695324994789700720601680257658265 0D−02	1.05D+02	4.99D+02
8	11	7.60D+01	2.8388989904669690643236023471269727 1D+01	3.39D+09	5.96D+10
12	29	6.93D+02	−6.3547489956249107854075245050797404 9D+00	7.70D+10	3.14D+12
16	80	7.04D+03	−6.3349204628046840409640223991345142 9D+00	6.28D+11	1.26D+14
20	227	7.53D+04	−6.3348995948342805793284406017289216 0D+00	1.50D+12	2.25D+15
24	646	8.00D+05	−6.3348996117742764458171541578743755 3D+00	1.99D+12	3.00D+16
28	1842	8.45D+06	−6.3348996117794283593081694022618246 2D+00	2.32D+12	3.74D+17
32	5258	8.89D+07	−6.3348996117794286223536923945035588 0D+00	2.57D+12	4.30D+18

By part 1 of Theorem 2.5, $\{A_n\}$ satisfies (3.1) with $\sigma = 1$. Therefore, Theorem 2.6 applies and we have the following result:

Theorem 6.1 *Consider the convergent infinite product* $\prod_{n=1}^{\infty}(1+v_n)$ *with* $v_n = w(n)$, $w \in \tilde{\boldsymbol{A}}_0^{(-t/m,m)}$ *strictly,* $t \geq m+1$ *being an integer. Let* $A_0 = 0$ *and* $A_n = \prod_{k=1}^{n}(1+v_k)$ *and* $a_n = A_n - A_{n-1}$, $n = 1, 2, \ldots$. *Then*

$$A_{n-1} = S + na_n g(n), \quad g \in \tilde{\boldsymbol{A}}_0^{(0,m)} \text{ strictly.} \tag{6.10}$$

Therefore, we also have

$$A_n - S \sim \alpha n^{1-t/m} \quad \text{as } n \to \infty, \quad \text{for some } \alpha \neq 0. \tag{6.11}$$

The asymptotic equality in (6.11) shows a very slow convergence rate for the sequence $\{A_n\}$ of the partial products in the case being considered.

Table 23 Numerical results for Example 7.1 using $R_l = l+1$, $l = 0, 1, \ldots$. Note that $S = 2/\pi$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	1.78D−01	1.78D−01	1.00D+00	1.18D+00
4	5	4.64D−02	6.53D−03	8.81D+01	9.33D+01
8	9	2.67D−02	3.75D−06	1.11D+04	1.14D+04
12	13	1.87D−02	3.16D−10	1.56D+06	1.60D+06
16	17	1.44D−02	7.34D−15	2.29D+08	2.33D+08
20	21	1.17D−02	6.56D−20	3.44D+10	3.49D+10
24	25	9.85D−03	1.95D−21	5.26D+12	5.33D+12
28	29	8.51D−03	4.89D−19	8.15D+14	8.24D+14
32	33	7.49D−03	6.42D−17	1.27D+17	1.29D+17

Table 24 Numerical results for Example 7.1, where the R_l are chosen using GPS with $\tau = 1.3$. Note that $S = 2/\pi$

n	R_n	$\lvert A_{R_n} - S\rvert/\lvert S\rvert$	$\lvert\bar{A}_n^{(0)} - S\rvert/\lvert S\rvert$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	1.78D−01	1.78D−01	1.00D+00	1.18D+00
4	5	4.64D−02	6.53D−03	8.81D+01	9.33D+01
8	11	2.20D−02	2.62D−06	2.92D+03	3.02D+03
12	29	8.51D−03	1.93D−11	3.30D+03	3.36D+03
16	80	3.11D−03	2.41D−18	2.74D+03	2.76D+03
20	227	1.10D−03	9.18D−27	2.09D+03	2.09D+03
24	646	3.87D−04	2.22D−30	1.90D+03	1.90D+03
28	1842	1.36D−04	2.38D−30	1.82D+03	1.82D+03
32	5258	4.75D−05	1.81D−29	1.78D+03	1.78D+03

Before closing, we mention that acceleration of the convergence of infinite products $\prod_{n=1}^{\infty}(1 + v_n)$ with $\{v_n\} \in \tilde{\mathbf{b}}^{(1)} = \mathbf{b}^{(1)}$ was first considered by Cohen and Levin [6], who use a method that is in the spirit of the d transformation.

7 Numerical examples II

We have applied the $\tilde{d}^{(m)}$ transformation to infinite products $\prod_{n=1}^{\infty}(1 + v_n)$ with $v_n = w(n) \in \tilde{\mathbf{A}}_0^{(-t/m,m)}$ for various values of $m \geq 1$ and verified that it is an effective convergence accelerator. We discuss one example with $m = 1$, for which S is known, and one example with $m = 2$, for which S is not known.

Example 7.1 Let $v_n = -z^2/n^2$, $n = 1, 2, \ldots$. Therefore, $m = 1$. It is known that $\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$. Here, we show the numerical results obtained by letting

Table 25 Numerical results for Example 7.2 using $R_l = l + 1$, $l = 0, 1, \ldots$. Note that the limit S is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.00D+00	2.00000000000000000000000000000000000D+00	1.00D+00	2.00D+00
4	5	3.96D+00	1.35479942920362927909756560802876967D+00	2.68D+02	9.27D+02
8	9	4.82D+00	−1.57042116160622622622411746352944801D+01	6.43D+06	2.80D+07
12	13	5.35D+00	9.53022999002732270167986334118038986D+00	3.05D+09	1.50D+10
16	17	5.71D+00	9.20517119198173662410564570330512235D+00	3.61D+12	1.91D+13
20	21	5.98D+00	9.20093271469623484626893604230623342D+00	4.70D+15	2.62D+16
24	25	6.19D+00	9.20090135744934917522373022100561298D+00	6.32D+18	3.68D+19
28	29	6.37D+00	9.20090121361950832779877985491595492D+00	8.69D+21	5.22D+22
32	33	6.51D+00	9.20090126648125570503784378346976091D+00	1.21D+25	7.48D+25

Table 26 Numerical results for Example 7.2, where the R_l are chosen using GPS with $\tau = 1.3$. Note that the limit S is not known

n	R_n	A_{R_n}	$\bar{A}_n^{(0)}$	$\Gamma_n^{(0)}$	$\Lambda_n^{(0)}$
0	1	2.00D+00	2.00000000000000000000000000000000D+00	1.00D+00	2.00D+00
4	5	3.96D+00	1.35479942920362927909756560802876967D+00	2.68D+02	9.27D+02
8	11	5.11D+00	−6.33525823410437784195106051773687585D+01	7.84D+06	3.33D+07
12	29	6.37D+00	9.25531467665752588697176832427936420D+00	6.64D+06	3.28D+07
16	80	7.36D+00	9.20093404854746290206845604435639641D+00	1.23D+07	7.46D+07
20	227	8.06D+00	9.20090121511720760645832075667746101D+00	1.35D+07	9.54D+07
24	646	8.50D+00	9.20090121315935366161482143819303662D+00	1.37D+07	1.08D+08
28	1842	8.78D+00	9.20090121315934117116570190908213652D+00	1.42D+07	1.19D+08
32	5258	8.95D+00	9.20090121315934117115672682505231045D+00	1.45D+07	1.26D+08

$z = 1/2$, for which we have

$$S = \frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right).$$

Table 23 contains results obtained by choosing $R_l = l + 1$, $l = 0, 1, \ldots$. In Table 24, we present results obtained by choosing the R_l using GPS with $\tau = 1.3$.

Example 7.2 Let $v_n = n^{-3/2}$, $n = 1, 2, \ldots$. Therefore, $m = 2$. In this case, S is not known. Table 25 contains results obtained by choosing $R_l = l + 1$, $l = 0, 1, \ldots$. In Table 26, we present results obtained by choosing the R_l using GPS with $\tau = 1.3$.

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