Calcolo (2021) 58:36
https://doi.org/10.1007/s10092-021-00414-9

Exactness and convergence properties of some recent numerical quadrature formulas for supersingular integrals of periodic functions

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Received: 28 October 2020 / Revised: 7 April 2021 / Accepted: 12 April 2021

Abstract

In a recent work, we developed three new compact numerical quadrature formulas for finite-range periodic supersingular integrals $I[f] = \ddagger\!\!\int_a^b f(x)\,dx$, where $f(x) = g(x)/(x-t)^3$, assuming that $g \in C^\infty[a,b]$ and $f(x)$ is T-periodic, $T = b - a$. With $h = T/n$, these numerical quadrature formulas read

$$\widehat{T}_n^{(0)}[f] = h\sum_{j=1}^{n-1} f(t+jh) - \frac{\pi^2}{3}\,g'(t)\,h^{-1} + \frac{1}{6}\,g'''(t)\,h,$$

$$\widehat{T}_n^{(1)}[f] = h\sum_{j=1}^{n} f(t+jh-h/2) - \pi^2\,g'(t)\,h^{-1},$$

$$\widehat{T}_n^{(2)}[f] = 2h\sum_{j=1}^{n} f(t+jh-h/2) - \frac{h}{2}\sum_{j=1}^{2n} f(t+jh/2-h/4).$$

We also showed that these formulas have spectral accuracy; that is,

$$\widehat{T}_n^{(s)}[f] - I[f] = o(n^{-\mu}) \quad \text{as } n \to \infty \quad \forall \mu > 0.$$

In the present work, we continue our study of these formulas for the special case in which $f(x) = \frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}}\,u(x)$, where $u(x)$ is in $C^\infty(\mathbb{R})$ and is T-periodic. Actually, we prove that $\widehat{T}_n^{(s)}[f]$, $s = 0, 1, 2$, are exact for a class of singular integrals involving T-periodic trigonometric polynomials of degree at most $n-1$; that is,

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Published online: 13 July 2021

$$\widehat{T}_n^{(s)}[f] = I[f] \quad \text{when } f(x) = \frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}} \sum_{m=-(n-1)}^{n-1} c_m \exp(\mathrm{i}2m\pi x/T).$$

We also prove that, when $u(z)$ is analytic in a strip $|\mathrm{Im}\, z| < \sigma$ of the complex z-plane, the errors in all three $\widehat{T}_n^{(s)}[f]$ are $O(e^{-2n\pi\sigma/T})$ as $n \to \infty$, for all practical purposes.

Keywords Hadamard Finite Part · Supersingular integrals · Numerical quadrature · Trapezoidal rule · Periodic integrands

Mathematics Subject Classification 41A55 · 65B15 · 65D30 · 65D32

1 Introduction and background

Let

$$I[f] = \ne\!\!\!\!\!\!\int_a^b f(x)\,dx, \quad f(x) = \frac{g(x)}{(x-t)^3}, \quad g \in C^\infty[a,b], \quad t \in (a,b) \text{ fixed.} \tag{1.1}$$

$\ne\!\!\!\!\!\int_a^b f(x)\,dx$ denotes the *Hadamard Finite Part (HFP)* of the *supersingular* integral $\int_a^b f(x)\,dx$, which does not exist in the regular sense due to the term $(x-t)^{-3}$.

In the recent work [6], the author developed three new trapezoidal-like numerical quadrature formulas $\widehat{T}_n^{(s)}[f]$, $s = 0, 1, 2$, for the HFP integrals in (1.1) that have excellent convergence properties for functions $f(x)$ that are such that

$$f(x) \;\; T\text{-periodic}, \quad f \in C^\infty(\mathbb{R}_t), \quad T = b-a, \quad \mathbb{R}_t = \mathbb{R} \setminus \{t \pm kT\}_{k=0}^{\infty}. \tag{1.2}$$

With $h = T/n$, these formulas read

$$\widehat{T}_n^{(0)}[f] = h\sum_{j=1}^{n-1} f(t+jh) - \frac{\pi^2}{3}\, g'(t)\, h^{-1} + \frac{1}{6}\, g'''(t)\, h, \tag{1.3}$$

$$\widehat{T}_n^{(1)}[f] = h\sum_{j=1}^{n} f(t+jh-h/2) - \pi^2\, g'(t)\, h^{-1}, \tag{1.4}$$

$$\widehat{T}_n^{(2)}[f] = 2h\sum_{j=1}^{n} f(t+jh-h/2) - \frac{h}{2}\sum_{j=1}^{2n} f(t+jh/2-h/4). \tag{1.5}$$

Theorem 5.1 in [6] states that, provided $f(x)$ is as in (1.1)–(1.2), $\widehat{T}_n^{(s)}[f] \to I[f]$ as $n \to \infty$ with spectral accuracy; that is,

$$\widehat{T}_n^{(s)}[f] - I[f] = o(n^{-\mu}) \quad \text{as } n \to \infty \quad \forall \mu > 0. \tag{1.6}$$

These quadrature formulas are obtained by manipulating a generalization of the Euler–Maclaurin expansion derived by the author in [4, Theorem 2.3], which we give in the next theorem.

Theorem 1.1 *Let $u(x)$ be such that $u \in C^\infty(a,b)$ and has the asymptotic expansions*

$$u(x) \sim K(x-a)^{-1} + \sum_{s=0}^{\infty} c_s\,(x-a)^{\gamma_s} \quad \text{as } x \to a+,$$

$$u(x) \sim L(b-x)^{-1} + \sum_{s=0}^{\infty} d_s\,(b-x)^{\delta_s} \quad \text{as } x \to b-,$$

$$\gamma_s \neq -1 \quad \forall s; \quad \mathrm{Re}\,\gamma_0 \leq \mathrm{Re}\,\gamma_1 \leq \mathrm{Re}\,\gamma_2 \leq \cdots; \quad \lim_{s\to\infty} \mathrm{Re}\,\gamma_s = +\infty,$$

$$\delta_s \neq -1 \quad \forall s; \quad \mathrm{Re}\,\delta_0 \leq \mathrm{Re}\,\delta_1 \leq \mathrm{Re}\,\delta_2 \leq \cdots; \quad \lim_{s\to\infty} \mathrm{Re}\,\delta_s = +\infty.$$

(The γ_s are distinct and arbitrary and so are the δ_s.) Assume also that these asymptotic expansions can be differentiated infinitely many times. Let also $h = (b-a)/n$, for $n = 1,2,\ldots$. Then, as $n \to \infty$, *there holds*

$$h\sum_{j=1}^{n-1} u(a+jh) \sim \neq\!\!\!\!\!\!\int_a^b u(x)\,dx + K(C - \log h) + \sum_{\substack{s=0 \\ \gamma_s \notin \{2,4,6,\ldots\}}}^{\infty} c_s\,\zeta(-\gamma_s)\,h^{\gamma_s+1}$$
$$+ L(C - \log h) + \sum_{\substack{s=0 \\ \delta_s \notin \{2,4,6,\ldots\}}}^{\infty} d_s\,\zeta(-\delta_s)\,h^{\delta_s+1}.$$

Here $\zeta(z)$ is the Riemann Zeta function and $C = 0.577\cdots$ is Euler's constant.

Note. When $K = L = 0$ and $\mathrm{Re}\,\gamma_0 > -1$, $\mathrm{Re}\,\delta_0 > -1$, $\int_a^b u(x)\,dx$ exists as a regular integral, hence $\neq\!\!\!\!\!\!\int_a^b u(x)\,dx = \int_a^b u(x)\,dx$. Otherwise, $\int_a^b u(x)\,dx$ is not defined in the regular sense, but its HFP $\neq\!\!\!\!\!\!\int_a^b u(x)\,dx$ is well-defined.

Supersingular integrals arise in different areas of science and engineering, and several numerical quadrature formulas for computing them exist in the literature. We do not intend to review them here; instead, we refer the reader to the bibliography of [6] for some of the related literature.

The main purpose of this work is two-fold: (i) to explore the exactness properties of the quadrature formulas $\widehat{T}_n^{(s)}[f]$ and (ii) to expand on the convergence properties of the $\widehat{T}_n^{(s)}[f]$ when $f(z)$ is T-periodic and analytic in a strip of the complex z-plane that includes the real axis, with poles of order three at $x = t + kT$, $k = 0, \pm 1, \pm 2, \ldots$; we aim at improving and refining (1.6) considerably.

The integrands we will be working with in the sequel are of the special form

$$f(x) = G(t,x)u(x), \quad G(t,x) \equiv \frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}}, \quad u \in C^\infty[a,b]. \tag{1.7}$$

Such integrands arise naturally when computing Cauchy transforms on the unit circle, for example. Throughout this work, we treat t as a fixed parameter and not as a variable.

The paper is organized as follows: In Sect. 2, we provide the statements of three theorems that concern the exactness properties of the quadrature formulas $\widehat{T}_n^{(s)}[f]$. Of these, Theorem 2.2 provides the eigenvalues λ_m and eigenfunctions $e_m(x)$ of the kernel $G(t,x)$, in the sense that

$$\,\,\,\,\,\,\,\,\,\,\,\,\neq\!\!\!\!\!\!\int_0^T G(t,x)e_m(x)\,dx = \lambda_m e_m(t), \quad m = 0, \pm 1, \pm 2, \ldots .$$

The proof of this theorem is given Sect. 3. Theorems 2.3 and 2.4 concern the action of the quadrature formulas $\widehat{T}_n^{(s)}[f]$ on the eigenfunctions of $G(t,x)$ and show that they preserve some of the eigenvalues. The proofs of these theorems are given in Sect. 4. In Sect. 5, we develop the subject of the convergence of the quadrature formulas as they are applied to T-periodic integrands $f(x) = G(t,x)u(x)$ when $u(z)$ is also analytic in a strip of the complex z-plane that contains the real axis. The main result of this development is stated as Theorem 5.2, whose proof is given in Sect. 6.

2 Exactness property of the $\widehat{T}_n^{(s)}[f]$

It is well-known that the trapezoidal rule for regular integrals has an interesting exactness property, as stated in Theorem 2.1:

Theorem 2.1 *Let $I[f] = \int_a^b f(x)\,dx$ be a regular integral. Then the trapezoidal rule approximation for $I[f]$, namely,*

$$Q_n[f] = h\left[\frac{1}{2}f(a) + \sum_{j=1}^{n-1} f(a+jh) + \frac{1}{2}f(b)\right]; \quad h = \frac{b-a}{n}, \quad n \text{ integer}, \tag{2.1}$$

is exact when $f(x)$ is a trigonometric polynomial of degree at most $n-1$ with period $T = b-a$. That is,

$$Q_n[f] = I[f] \quad \forall f(x) = \sum_{m=-(n-1)}^{n-1} c_m e^{\mathrm{i}\,2m\pi x/T}. \tag{2.2}$$

We showed in [5, Theorems 5.1 and 10.1] that the numerical quadrature formulas developed there for periodic *Cauchy Principal Value* integrals and *hypersingular* integrals, when applied to $\neq\!\!\!\!\!\int_a^b \cot\frac{\pi(x-t)}{T}u(x)\,dx$ and $\neq\!\!\!\!\!\int_a^b \csc^2\frac{\pi(x-t)}{T}u(x)\,dx$, respectively, also enjoy interesting exactness properties when $u(x)$ is a trigonometric polynomial of period $T = b - a$.

Here we show that each of the numerical quadrature formulas $\widehat{T}_n^{(s)}[f]$ given in (1.3)–(1.5) for supersingular integrals $I[f]$ with T-periodic $f(x)$, enjoy similar exactness properties as described in Theorems 2.2 and 2.3, which form two of the main results of this work.

Theorem 2.2 *Let $G(t,x)$ be as in (1.7). With $T = b - a$, let*

$$e_m(x) = e^{\mathrm{i}2m\pi x/T}, \quad f_m(x) = G(t,x)e_m(x), \quad m = 0, \pm 1, \pm 2, \dots. \tag{2.3}$$

Then the supersingular integral $I[f_m] = \,=\!\!\!\!\!\!\int_a^b f_m(x)\,dx$ satisfies

$$I[f_m] = -\mathrm{i}\, sgn(m) 2Tm^2 e_m(t), \quad m = 0, \pm 1, \pm 2, \dots. \tag{2.4}$$

[*Note that $f_m(x)$ is T-periodic and has a supersingularity of the form $(x-t)^{-3}$ at $x = t$.*]

Remark Observe that this theorem actually states that $e_m(x)$ in (2.3) are actually eigenfunctions of the kernel $G(t,x) = \cos \frac{\pi(x-t)}{T} / \sin^3 \frac{\pi(x-t)}{T}$ with corresponding eigenvalues $\lambda_m = -\mathrm{i}\,\mathrm{sgn}(m) 2Tm^2$, $m = 0, \pm 1, \pm 2, \dots$, which we alluded to in Sect. 1.

Theorem 2.3 *With $e_m(x)$ and $f_m(x)$ as in Theorem 2.2, the quadrature formulas $\widehat{T}_n^{(s)}[f_m]$ satisfy the following*:

$$\widehat{T}_n^{(0)}[f_m] = \mathrm{i}\,\frac{T}{n}\left(B_{m,n} - \frac{2}{3}mn^2 - \frac{4}{3}m^3\right) e_m(t), \tag{2.5}$$

$$\widehat{T}_n^{(1)}[f_m] = \mathrm{i}\,\frac{T}{n}\left([B_{m,2n} - B_{m,n}] - 2mn^2\right) e_m(t), \tag{2.6}$$

$$\widehat{T}_n^{(2)}[f_m] = \mathrm{i}\,\frac{T}{n}\left(2[B_{m,2n} - B_{m,n}] - \frac{1}{2}[B_{m,4n} - B_{m,2n}]\right) e_m(t), \tag{2.7}$$

where $B_{m,n}$ are defined as follows:

1. For $m = 0$, we have $B_{0,n} = 0$. For arbitrary m, there holds $B_{-m,n} = -B_{m,n}$.
2. Given $m \geq 0$, let k and r be (unique) integers, $k \geq 0$ and $0 \leq r \leq n-1$, such that $m = kn + r$. Then

$$B_{m,n} = B_{kn+r,n} = B_{r,n} = \frac{2}{3}\,rn^2 - 2r^2 n + \frac{4}{3}\,r^3. \tag{2.8}$$

3. Thus, $\left|B_{m,n}\right| \leq \max_{0 \leq i \leq n-1} \left|B_{i,n}\right|$ independent of m, hence $\{B_{m,n}\}_{m=-\infty}^{\infty}$ is a bounded sequence for each fixed n.

Theorem 2.4 *All three quadrature formulas $\widehat{T}_n^{(s)}[f]$ possess the exactness property that*

$$\widehat{T}_n^{(s)}[f_m] = I[f_m], \quad m = 0, \pm 1, \dots, \pm(n-1), \tag{2.9}$$

hence that

$$\widehat{T}_n^{(s)}[f] = I[f], \quad f(x) = \frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}}\,u(x) \quad \forall\, u(x) = \sum_{m=-(n-1)}^{n-1} c_m e^{\mathrm{i}\,2m\pi x/T}. \tag{2.10}$$

We provide the proofs of these theorems in the next two sections.

3 Proof of Theorem 2.2

We start by noting that

$$I[f_m] = \left[⨎_a^b \frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}} e^{\mathrm{i}\,2m\pi(x-t)/T} dx \right] e^{\mathrm{i}\,2m\pi t/T}.$$

Making the variable transformation $y = 2\pi(x-t)/T$ in the integral inside the square brackets, and using the fact that the transformed integrand is 2π-periodic, we obtain

$$I[f_m] = \frac{T}{2\pi} A_m e^{\mathrm{i}\,2m\pi t/T}, \quad A_m = ⨎_{-\pi}^{\pi} \frac{\cos(\frac{1}{2}y)}{\sin^3(\frac{1}{2}y)} e^{\mathrm{i}my}\, dy. \tag{3.1}$$

Next,

$$A_m = ⨎_{-\pi}^{\pi} \frac{\cos(\frac{1}{2}y)}{\sin^3(\frac{1}{2}y)} \cos(my)\, dy + \mathrm{i} ⨎_{-\pi}^{\pi} \frac{\cos(\frac{1}{2}y)}{\sin^3(\frac{1}{2}y)} \sin(my)\, dy,$$

and since $⨎_{-\pi}^{\pi} [\cos(\frac{1}{2}y)/\sin^3(\frac{1}{2}y)] \cos(my)\, dy = 0$ due to its integrand being odd, it follows that

$$A_m = \mathrm{i} ⨎_{-\pi}^{\pi} \frac{\cos(\frac{1}{2}y)}{\sin^3(\frac{1}{2}y)} \sin(my)\, dy \quad \Rightarrow \quad A_0 = 0, \quad A_{-m} = -A_m. \tag{3.2}$$

Therefore, it is sufficient to study A_m only for nonnegative m, and this is what we do in the sequel.

Now,

$$\begin{aligned} \sin[(m+1)y] + \sin[(m-1)y] &= 2\sin(my)\cos y \\ &= 2\left[1 - 2\sin^2\left(\tfrac{1}{2}y\right)\right] \sin(my), \end{aligned}$$

from which, by multiplying by $\cos(\frac{1}{2}y)/\sin^3(\frac{1}{2}y)$, we obtain the identity

$$\frac{\cos\left(\frac{1}{2}y\right)}{\sin^3\left(\frac{1}{2}y\right)}\big(\sin[(m+1)y]-2\sin(my)+\sin[(m-1)y]\big) \\ =-4\cot\left(\frac{1}{2}y\right)\,\sin(my). \tag{3.3}$$

Upon integrating both sides of this identity over $(-\pi,\pi)$ and invoking (3.2), we obtain

$$A_{m+1}-2A_m+A_{m-1}=-4\mathrm{i}\int_{-\pi}^{\pi}\cot(\tfrac{1}{2}y)\sin(my)\,dy. \tag{3.4}$$

[Note that the integral on the right-hand side of (3.4) is defined in the regular sense.] By making the variable transformation $y=2z$ in this integral, and invoking [2, p. 391, formula 3.612(7)], we obtain

$$\int_{-\pi}^{\pi}\cot(\tfrac{1}{2}y)\sin(my)\,dy=4\int_0^{\pi/2}\cos z\,\frac{\sin(2mz)}{\sin z}\,dz=2\pi,\quad m=1,2,\ldots.$$

Substituting this in (3.4), we obtain the following linear nonhomogeneous three-term recursion relation for the A_m:

$$A_{m+1}-2A_m+A_{m-1}=-8\pi\mathrm{i},\quad m=1,2,\ldots.$$

It is easy to see that the general solution of this recursion relation for A_m is of the form

$$A_m=\alpha+\beta m-\mathrm{i}4\pi m^2,\quad \alpha \text{ and } \beta \text{ constants to be determined.}$$

First, $A_0=0$ by (3.2); this gives $\alpha=0$. Next, letting $m=1$ in the integral representation of A_m in (3.2), and simplifying the integrand, we obtain

$$A_1=\mathrm{i}\,⨎_{-\pi}^{\pi}\frac{\cos(\frac{1}{2}y)}{\sin^3(\frac{1}{2}y)}\sin y\,dy=2\mathrm{i}\,⨎_{-\pi}^{\pi}\left[\frac{1}{\sin^2(\frac{1}{2}y)}-1\right]dy,$$

which, by the fact that (see [5, Appendix A], for example) $⨎_{-\pi}^{\pi}\csc^2(\frac{1}{2}y)\,dy=0$, gives $A_1=-4\pi\mathrm{i}$, which in turn implies $\beta=0$. Consequently, taking into account that $A_{-m}=-A_m$, we have

$$A_m=-\mathrm{i}\,\mathrm{sgn}(m)\,4\pi m^2,\quad m=0,\pm1,\pm2,\ldots,$$

which, upon substituting into (3.1), gives (2.4).

4 Proofs of Theorems 2.3 and 2.4

4.1 Preliminaries

By the fact that $g(x) = (x-t)^3 f(x)$ in (1.1), we realize that we must first address the issue of determining $g(x)$ and its first three derivatives at $x = t$ when $f(x)$ is of the form,

$$f(x) = \frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}}\,u(x) \quad \Rightarrow \quad g(x) = (x-t)^3\frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}}\,u(x). \tag{4.1}$$

We achieve this by expanding $g(x)$ in a Taylor series about $x = t$. We start by realizing that

$$\frac{\cos z}{\sin^3 z} = \frac{1}{z^3}\frac{1-\frac{1}{2}z^2+O\big(z^4\big)}{1-\frac{1}{2}z^2+O\big(z^4\big)} = \frac{1}{z^3}\big[1+O\big(z^4\big)\big] \quad \text{as } z \to 0.$$

Using this in (4.1) and also expanding $u(x)$ about $x = t$, we obtain

$$g(x) = \left(\frac{T}{\pi}\right)^3\left[\sum_{i=0}^{3}\frac{u^{(i)}(t)}{i!}(x-t)^i + O\big((x-t)^4\big)\right] \quad \text{as } x \to t,$$

which implies that

$$g^{(i)}(t) = \left(\frac{T}{\pi}\right)^3 u^{(i)}(t), \quad i = 0,1,2,3. \tag{4.2}$$

Next, letting

$$\widetilde{T}_n[f] = h\sum_{j=1}^{n-1} f(t+jh), \tag{4.3}$$

we rewrite (1.3)–(1.5) in the form (see [6, Section 4])

$$\widehat{T}_n^{(0)}[f] = \widetilde{T}_n[f] - \frac{\pi^2}{3}\,g'(t)\,h^{-1} + \frac{1}{6}\,g'''(t)\,h, \tag{4.4}$$

$$\widehat{T}_n^{(1)}[f] = (2\widetilde{T}_{2n}[f] - \widetilde{T}_n[f]) - \pi^2\,g'(t)\,h^{-1}, \tag{4.5}$$

$$\widehat{T}_n^{(2)}[f] = 2(2\widetilde{T}_{2n}[f] - \widetilde{T}_n[f]) - (2\widetilde{T}_{4n}[f] - \widetilde{T}_{2n}[f]). \tag{4.6}$$

This suggests that we can unify and shorten the proofs for the three $\widehat{T}_n^{(s)}[f_m]$ since we only have to analyze $\widetilde{T}_n[f_m]$ in detail. We do this in Theorem 4.1 that follows.

4.2 Analysis of $\widetilde{T}_n[f_m]$

Theorem 4.1 *$\widetilde{T}_n[f_m]$ satisfies*

$$\widetilde{T}_n[f_m] = \left(\mathrm{i}\,\frac{T}{n}\,B_{m,n}\right)e_m(t), \quad m = 0, \pm 1, \pm 2, \dots, \tag{4.7}$$

where $B_{m,n}$ has the following properties:

$$B_{-m,n} = -B_{m,n}\ \forall m \quad \Rightarrow \quad B_{0,n} = 0, \tag{4.8}$$

$$B_{m,n} = \frac{2}{3}mn^2 - 2m^2n + \frac{4}{3}m^3, \quad m = 1, \dots n-1, \tag{4.9}$$

$$B_{m,n} = sgn(m)B_{kn+r,n} = sgn(m)B_{r,n}\ \forall m; \quad k \ge 0,\ r \in \{0, 1, \dots, n-1\}, \tag{4.10}$$

where k and r are unique integers for which $|m| = kn + r$.

Proof We start by observing that, by (4.3),

$$\widetilde{T}_n[f_m] = \mathrm{i}\,\frac{T}{n}B_{m,n}, \quad B_{m,n} = -\mathrm{i}\sum_{j=1}^{n-1}\frac{\cos\left(\frac{1}{2}y_j\right)}{\sin^3\left(\frac{1}{2}y_j\right)}\,e^{\mathrm{i}my_j}, \quad y_j = \frac{2j\pi}{n}, \quad j = 1, 2, \dots. \tag{4.11}$$

Now

$$B_{m,n} = -\mathrm{i}\sum_{j=1}^{n-1}\frac{\cos\left(\frac{1}{2}y_j\right)}{\sin^3\left(\frac{1}{2}y_j\right)}\cos(my_j) + \sum_{j=1}^{n-1}\frac{\cos\left(\frac{1}{2}y_j\right)}{\sin^3\left(\frac{1}{2}y_j\right)}\sin(my_j).$$

Because $y_{n-j} = 2\pi - y_j$, $j = 1, \dots, n-1$, we have

$$\frac{\cos\left(\frac{1}{2}y_{n-j}\right)}{\sin^3\left(\frac{1}{2}y_{n-j}\right)}\cos(my_{n-j}) = -\frac{\cos\left(\frac{1}{2}y_j\right)}{\sin^3\left(\frac{1}{2}y_j\right)}\cos(my_j), \quad j = 1, \dots, n-1,$$

and since $\sum_{j=1}^{n-1} w_{n-j} = \sum_{j=1}^{n-1} w_j$, we have

$$\sum_{j=1}^{n-1}\frac{\cos\left(\frac{1}{2}y_j\right)}{\sin^3\left(\frac{1}{2}y_j\right)}\cos(my_j) = 0.$$

As a result,

$$B_{m,n} = \sum_{j=1}^{n-1} \frac{\cos\left(\frac{1}{2}y_j\right)}{\sin^3\left(\frac{1}{2}y_j\right)} \sin(my_j) \quad \Rightarrow \quad B_{0,n} = 0, \quad B_{-m,n} = -B_{m,n}, \quad (4.12)$$

hence (4.8) is proved. Therefore, it is sufficient to study $B_{m,n}$ only for positive m.

Next, for every $m \geq 0$, there exist unique integers k and r, $k \geq 0$ and $0 \leq r \leq n-1$, such that $m = nk + r$. (Thus, $k = 0$ and $r = m$ for $0 \leq m \leq n-1$, while $k = 1$ and $r = 0$ for $m = n$.) By the fact that

$$\sin[(kn + r)y_j] = \sin(2kj\pi + ry_j) = \sin(ry_j), \quad r = 0, \ldots, n-1,$$

we realize that

$$B_{m,n} = B_{kn+r,n} = B_{r,n} \quad \text{when } m \geq 0, \quad (4.13)$$

which, upon combining with (4.12), results in (4.10). Thus, we need to concern ourselves only with $1 \leq m \leq n-1$ since $k = 0$ and $r = m$ in such a case, and this is what we do in the sequel.

We start by deriving a recursion relation for the $B_{m,n}$ analogous to that for the A_m given in (3.4). Replacing y in (3.3) by y_j and summing over j, we obtain

$$B_{m+1,n} - 2B_{m,n} + B_{m-1,n} = -4C_{m,n}, \quad C_{m,n} = \sum_{j=1}^{n-1} \cot\left(\tfrac{1}{2}y_j\right) \sin(my_j). \quad (4.14)$$

To determine $C_{m,n}$, we proceed as follows: First,

$$\sin\left(my \pm \tfrac{1}{2}y\right) = \sin(my)\cos\left(\tfrac{1}{2}y\right) \pm \cos(my)\sin\left(\tfrac{1}{2}y\right). \quad (4.15)$$

Dividing both sides of this identity by $\sin(\frac{1}{2}y)$, replacing y by y_j, and summing over j, we obtain

$$\sum_{j=1}^{n-1} \frac{\sin\left(my_j \pm \frac{1}{2}y_j\right)}{\sin\left(\frac{1}{2}y_j\right)} = C_{m,n} \pm \sum_{j=1}^{n-1} \cos(my_j). \quad (4.16)$$

Now,

$$\sum_{j=1}^{n-1} \cos(my_j) = \text{Re} \sum_{j=1}^{n-1} e^{\text{i}my_j} = \text{Re} \sum_{j=1}^{n-1} \left(e^{\text{i}2m\pi/n}\right)^j = -1, \quad m = 1, \ldots, n-1,$$

since $e^{\text{i}2m\pi/n} \neq 1$ for $m = 1, \ldots, n-1$. Upon also defining

$$D_{k,n} = \sum_{j=1}^{n-1} \frac{\sin\left(ky_j - \frac{1}{2}y_j\right)}{\sin\left(\frac{1}{2}y_j\right)},$$

(4.16) gives the equalities

$$D_{m,n} = C_{m,n} + 1 \quad \text{and} \quad D_{m+1,n} = C_{m,n} - 1, \quad m = 1, \dots, n-1.$$

Eliminating $C_{m,n}$, we obtain

$$D_{m+1,n} = D_{m,n} - 2, \quad m = 1, \dots, n-1,$$

which, upon realizing that $D_{1,n} = n - 1$, gives

$$D_{m,n} = n - 2m + 1 \quad \Rightarrow \quad C_{m,n} = n - 2m, \quad m = 1, \dots, n-1.$$

As a result, (4.14) becomes

$$B_{m+1,n} - 2B_{m,n} + B_{m-1,n} = 8m - 4n, \quad m = 1, \dots, n-1. \tag{4.17}$$

It is easy to see that the general solution of this recursion relation for $B_{m,n}$ is of the form

$$B_{m,n} = \alpha + \beta m - 2nm^2 + \frac{4}{3}m^3, \quad m \geq 1, \tag{4.18}$$

α and β being constants to be determined. They can be obtained by invoking the values of $B_{1,n}$ and $B_{2,n}$.

We start with $B_{1,n}$. Letting $m = 1$ in (4.12) and simplifying, we obtain

$$B_{1,n} = 2\sum_{j=1}^{n-1}\left(\frac{1}{\sin^2\left(\frac{1}{2}y_j\right)} - 1\right) = 2(L_n - n + 1), \quad L_n = \sum_{j=1}^{n-1}\frac{1}{\sin^2\left(\frac{1}{2}y_j\right)}. \tag{4.19}$$

To determine L_n, we proceed as follows: We first express L_n in the form

$$\begin{aligned} L_n &= \sum_{j=1}^{n-1}\frac{1}{1-\eta_j^2} = \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{1}{1-\eta_j} + \frac{1}{1+\eta_j}\right) \\ &= \sum_{j=1}^{n-1}\frac{1}{1-\eta_j}, \quad \eta_j = \cos\left(\frac{j\pi}{n}\right). \end{aligned} \tag{4.20}$$

Here we have invoked $\eta_{n-j} = -\eta_j$ and $\sum_{j=1}^{n-1} w_j = \sum_{j=1}^{n-1} w_{n-j}$. Now, $\eta_1, \dots, \eta_{n-1}$ are the points of extremum of the nth Chebyshev polynomial $T_n(z)$ in $(-1, 1)$, hence the zeros of its derivative $T_n'(z)$. Thus,

$$\frac{T_n''(z)}{T_n'(z)} = \sum_{j=1}^{n-1}\frac{1}{z-\eta_j},$$

hence

$$L_n = \frac{T_n''(1)}{T_n'(1)} = \frac{n^2 - 1}{3}. \tag{4.21}$$

(See, [3, p. 38], for example.) Consequently,

$$B_{1,n} = 2\left(\frac{n^2 - 1}{3} - n + 1\right).$$

As for $B_{2,n}$, letting $m = 1$ in (4.17) and recalling that $B_{0,n} = 0$, we obtain

$$B_{2,n} = 2B_{1,n} + 8 - 4n.$$

Substituting these values of $B_{1,n}$ and $B_{2,n}$ into (4.18) (with $m = 1$ and $m = 2$ there), we obtain $\alpha = 0$ and $\beta = 2n^2/3$, hence (4.9). □

4.3 Completion of proofs

With Theorem 4.1 available, we can now complete the proof of Theorem 2.3. Since $u(x) = e_m(x)$ when $f(x) = f_m(x)$, from (4.1) and (4.2), we have

$$g'(t) = \frac{T^3}{\pi^3} e_m'(t) = \mathrm{i}\, 2\frac{T^2}{\pi^2} m e_m(t) \quad \text{and} \quad g'''(t) = \frac{T^3}{\pi^3} e_m'''(t) = -\mathrm{i}\, 8m^3 e_m(t).$$

Substituting these in (4.4)–(4.6) and invoking also (4.7), we obtain (2.5)–(2.7). Finally, the expression given for $B_{m,n}$ in (2.8) is simply that in (4.9) proved in Theorem 4.1. This completes the proof of Theorem 2.3.

To complete the proof of Theorem 2.4, we only need to verify (2.9). We can achieve this by substituting (4.9) in (2.5)–(2.7) and comparing with (2.4). We leave the details to the reader.

5 Convergence property of the $\widehat{T}_n^{(s)}[f]$

It is well known that the trapezoidal rule $Q_n[f]$ in (2.1) converges exponentially in n when applied to *regular* integrals $I[f] = \int_a^b f(x)\,dx$ in case $f(z)$, as a function of the complex variable z, is analytic in a strip of the z-plane containing the real axis and is $(b-a)$-periodic in this strip. The following theorem from [1], addresses this fully:

Theorem 5.1 *Let $f(z)$ be analytic and periodic with period $T = b - a$ in the infinite strip $D_\sigma = \{z : |\mathrm{Im}\, z| < \sigma\}$ of the z-plane. Then*

$$\left|Q_n[f] - I[f]\right| \le TM(\tau)\frac{e^{-2n\pi\tau/T}}{1 - e^{-2n\pi\tau/T}} \quad \forall \tau \in (0, \sigma), \tag{5.1}$$

where

$$M(\tau) = \max_{x\in\mathbb{R}} \left|f(x+\mathrm{i}\tau)\right| + \max_{x\in\mathbb{R}} \left|f(x-\mathrm{i}\tau)\right|. \tag{5.2}$$

We showed in [7, Theorem 9] and in [5, Theorems 6.1 and 6.2] that the numerical quadrature formulas developed in these papers for periodic *Cauchy Principal Value* integrals and *hypersingular* integrals, when applied to $⨍_a^b \cot\frac{\pi(x-t)}{T}u(x)\,dx$ and $⨎_a^b \csc^2\frac{\pi(x-t)}{T}u(x)\,dx$, respectively, also enjoy similar convergence properties when $f(z)$ is T-periodic and has poles of order one and two, respectively, at the points $t+kT$, $k=0,\pm1,\pm2,\dots$, and is analytic in a strip of the z-plane containing the real axis.

Here we show that each of the numerical quadrature formulas $\widehat{T}_n^{(s)}[f]$ given in (1.3)–(1.5) for supersingular integrals $I[f]$ with T-periodic $f(z)$ enjoys similar convergence properties, as described in Theorem 5.2. The proof of this theorem is provided in Sect. 6.

Theorem 5.2 *Let the function $u(z)$ be analytic and periodic with period $T=b-a$ in the infinite strip $D_\sigma=\{z:\ |\mathrm{Im}\,z|<\sigma\}$ of the z-plane, and let*

$$f(x)=\frac{\cos\frac{\pi(x-t)}{T}}{\sin^3\frac{\pi(x-t)}{T}}\,u(x)\quad\text{and}\quad I[f]=⨏_a^b f(x)\,dx.$$

Define $E_n^{(s)}[f]=\widehat{T}_n^{(s)}[f]-I[f]$, $s=0,1,2$. Then

$$\left|E_n^{(0)}[f]\right|\le TM(\tau)\phi_n(\tau)\quad\forall\tau\in(0,\sigma), \tag{5.3}$$

$$\left|E_n^{(1)}[f]\right|\le TM(\tau)[\phi_n(\tau)+2\phi_{2n}(\tau)]\quad\forall\tau\in(0,\sigma), \tag{5.4}$$

$$\left|E_n^{(2)}[f]\right|\le TM(\tau)[2\phi_n(\tau)+5\phi_{2n}(\tau)+2\phi_{4n}(\tau)]\quad\forall\tau\in(0,\sigma). \tag{5.5}$$

where

$$M(\tau)=\max_{x\in\mathbb{R}}\left|F_1(x+\mathrm{i}\tau)\right|+\max_{x\in\mathbb{R}}\left|F_1(x-\mathrm{i}\tau)\right|,\quad \phi_n(\tau)=\frac{e^{-2n\pi\tau/T}}{1-e^{-2n\pi\tau/T}}. \tag{5.6}$$

Here $F_1(z)$ is T-periodic and analytic in the strip D_σ and is given as

$$F_1(z)=G(t,z)\left[u(z)-u(t)-\frac{T}{\pi}u'(t)\tan\frac{\pi(z-t)}{T}-\frac{T^2}{2\pi^2}u''(t)\sin^2\frac{\pi(z-t)}{T}\right].$$

Remark It is easy to see that, for all practical purposes, all three errors $E_n^{(s)}[f]$ are $O(e^{-2n\pi\sigma/T})$ as $n\to\infty$. Of course, this improves the convergence result in (1.6) significantly for the supersingular integrals considered here.

6 Proof of Theorem 5.2

We start by observing that, with $\widetilde{T}_n[f]$ as in (4.3) and $\widehat{T}_n^{(0)}[f]$ as in (4.4), we can reexpress $\widehat{T}_n^{(1)}[f]$ in (4.5) and $\widehat{T}_n^{(2)}[f]$ in (4.6) as follows:

$$\widehat{T}_n^{(1)}[f] = 2\widehat{T}_{2n}^{(0)}[f] - \widehat{T}_n^{(0)}[f], \tag{6.1}$$

$$\widehat{T}_n^{(2)}[f] = -2\widehat{T}_{4n}^{(0)} + 5\widehat{T}_{2n}^{(0)}[f] - 2\widehat{T}_n^{(0)}[f]. \tag{6.2}$$

As a result, we also have

$$E_n^{(1)}[f] = 2E_{2n}^{(0)}[f] - E_n^{(0)}[f], \tag{6.3}$$

$$E_n^{(2)}[f] = -2E_{4n}^{(0)}[f] + 5E_{2n}^{(0)}[f] - 2E_n^{(0)}[f]. \tag{6.4}$$

Clearly, this will help us unify the treatments of all three $\widehat{T}_n^{(s)}[f]$, once we treat $\widehat{T}_n^{(0)}[f]$.

Next, following [8], we let

$$u(z) = U_1(z) + U_2(z); \quad U_2(z) = u(t) + \frac{T}{\pi}u'(t)\tan\frac{\pi(z-t)}{T} + \frac{T^2}{2\pi^2}u''(t)\sin^2\frac{\pi(z-t)}{T},$$

and

$$f(z) = F_1(z) + F_2(z); \quad F_1(z) = G(t,z)\,U_1(z), \quad F_2(z) = G(t,z)\,U_2(z).$$

Therefore,

$$I[f] = I[F_1] + I[F_2] \quad \text{and} \quad \widehat{T}_n^{(0)}[f] = \widehat{T}_n^{(0)}[F_1] + \widehat{T}_n^{(0)}[F_2].$$

Since $U_1(z)$ and $U_2(z)$ are both T-periodic, so are $F_1(z)$ and $F_2(z)$. Expanding $U_2(z)$ about $z = t$ in a Taylor series, it is easy to verify that

$$\begin{aligned} &U_1(t) = U_1'(t) = U_1''(t) = 0, \quad U_1'''(t) = u'''(t) - \frac{2\pi^2}{T^2}u'(t) \\ &\Rightarrow \quad F_1(t) = \frac{T^3}{6\pi^3}U_1'''(t), \end{aligned} \tag{6.5}$$

which implies that $F_1(z)$ has no singularities in the strip D_σ and that $I[F_1]$ is a regular integral, to which Theorem 5.1 applies.

Let us now study $I[F_2]$ and $\widehat{T}_n^{(0)}[F_2]$. We have

$$I[F_2] = u(t)I_1 + \frac{T}{\pi}u'(t)I_2 + \frac{T^2}{2\pi^2}u''(t)I_3,$$

where by Theorems 2.2 and 2.3,

$$I_1 = I[G(t,\cdot)e_0] = 0 = \widehat{T}_n^{(0)}[G(t,\cdot)e_0]$$

and since $\sin^2\frac{\pi(x-t)}{T} = \frac{1}{4}[2e_0(x) - e_2(x)e_{-2}(t) - e_{-2}(x)e_2(t)]$,

$$I_3 = I\Big[G(t,\cdot)\sin^2\tfrac{\pi(\cdot-t)}{T}\Big] = 0 = \widehat{T}_n^{(0)}\Big[G(t,\cdot)\sin^2\tfrac{\pi(\cdot-t)}{T}\Big] \quad \forall\, n \geq 3.$$

Next, it is known that

$$I_2 = I[G(t,\cdot)\tan\tfrac{\pi(\cdot-t)}{T}] = \int\!\!\!\!\!\!=_a^b \frac{1}{\sin^2\frac{\pi(x-t)}{T}}\,dx = 0.$$

As for $\widehat{T}_n^{(0)}[G(t,\cdot)\tan\frac{\pi(\cdot-t)}{T}]$, by (4.2)–(4.4) and (4.19)–(4.21), and by the fact that $\tan z = z + \frac{1}{3}z^3 + O(z^5)$,

$$\widetilde{T}_n^{(0)}\left[G(t,\cdot)\tan\tfrac{\pi(\cdot-t)}{T}\right] = \frac{T}{n}L_n \quad \Rightarrow \quad \widehat{T}_n^{(0)}\left[G(t,\cdot)\tan\tfrac{\pi(\cdot-t)}{T}\right] = 0.$$

We have thus shown that $I[F_2] = 0 = \widehat{T}_n^{(0)}[F_2]$. We conclude that $I[f] = I[F_1]$ and $\widetilde{T}_n^{(0)}[f] = \widetilde{T}_n^{(0)}[F_1]$.

We now wish to show that $\widehat{T}_n^{(0)}[f] = Q_n[F_1]$, where $Q_n[F_1]$, the trapezoidal rule approximation for $I[F_1]$, is given as

$$Q_n[F_1] = h\sum_{j=0}^{n-1} F_1(t+jh) = \widetilde{T}_n^{(0)}[F_1] + hF_1(t)$$

since $F_1(z)$ is T-periodic. Therefore, by (4.2)–(4.4) and (6.5),

$$\widehat{T}_n^{(0)}[F_1] = Q_n[F_1] - hF_1(t) + \frac{T^3}{\pi^3}\left(-\frac{\pi^2}{3}U_1'(t)h^{-1} + \frac{1}{6}U_1'''(t)h\right) = Q_n[F_1].$$

Combining everything, we have shown that $\widehat{T}_n^{(0)}[f] - I[f] = Q_n[F_1] - I[F_1]$. We now complete the proof of (5.3) by applying Theorem 5.1 to $I[F_1]$. The proofs of (5.4) and (5.5) are immediate.

References

1. Davis, P.J.: On the numerical integration of periodic analytic functions. In: Langer, R.E. (ed.) On Numerical Approximation, University of Wisconsin Press, Madison, pp. 45–59 (1959)
2. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Academic Press, New York (2007)
3. Rivlin, T.J.: Chebyshev Poynomials: From Approximation Theory to Algebra and Number Theory, 2nd edn. Wiley, New York (1990)
4. Sidi, A.: Euler–Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities. Math. Comput. **81**, 2159–2173 (2012)
5. Sidi, A.: Compact numerical quadrature formulas for hypersingular integrals and integral equations. J. Sci. Comput. **54**, 145–176 (2013)
6. Sidi, A.: Unified compact numerical quadrature formulas for Hadamard finite parts of singular integrals of periodic functions. Calcolo **58**, 22 (2021)
7. Sidi, A., Israeli, M.: Quadrature methods for periodic singular and weakly singular Fredholm integral equations. J. Sci. Comput. **3**, 201–231 (1988). (**Originally appeared as Technical Report No. 384, Computer Science Dept., Technion–Israel Institute of Technology, (1985), and also as ICASE Report No. 86-50 (1986)**)
8. Yang, Chaoxia: A unified approach with spectral convergence for evaluation of hypersingular and supersingular integrals with a periodic kernel. J. Comput. Appl. Math. **239**, 322–332 (2013)