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Some biorthogonal polynomials arising in numerical analysis and approximation theory

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ABSTRACT

We survey some polynomials $\{P_n\}$ arising from convergence acceleration, and numerical integration, that satisfy "biorthogonality" conditions such as

$$\int_a^b P_n(x)\,\phi_j(x)\,w(x)\,dx = 0,$$

for appropriate functions $\{\phi_j\}$ and weights w. One example is $\phi_j(x) = (\log x)^j$, $0 \le j \le n-1$ on $[a,b] = [0,1]$. We discuss identities, asymptotics, positive quadratures, and zero distributions. We also list some open questions.

1. Introduction

In the traditional sense, biorthogonal polynomials involve two sequences of polynomials $\{p_n\}$ and $\{q_n\}$, as well as a linear functional L for which

$$L(p_j q_k) = \delta_{jk}.$$

As C. Brezinski notes [1, p. 104], in some form the idea goes back at least to Didon in 1869. However, in recent years, the notion of biorthogonal polynomials has been used in a much more general sense. Thus the term has been used [2] for polynomials that are orthogonal to some measures $\{\mu_j\}_{j=1}^n$:

$$\int p_n d\mu_j = 0,\ 1 \le j \le n, \tag{1.1}$$

or more specially to some functions. In the theory of random matrices [3], they have been defined by conditions such as

$$\iint p_j(x)\,q_k(y)\,w(x,y)\,dx\,dy = 0,\ \ j \ne k,$$

or by Cauchy (possibly singular) conditions such as [4,5]

$$\iint \frac{p_j(x)\,q_k(y)}{x \pm y}\,d\sigma_1(x)\,d\sigma_2(y) = 0,\ \ j \ne k,$$

where σ_1 and σ_2 are measures.

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The most general form we shall discuss in this paper is that defined by (1.1). However, much of our discussion focuses on more special polynomials, that have their origin in the work of the second author on convergence acceleration and numerical integration. Brezinski's book [1] is an excellent source for applications of various forms of biorthogonality in numerical analysis. A quite general setting for biorthogonal polynomials has also been studied by Iserles, Nørsett, and Saff in a series of papers [2,6,7], with interesting applications to transformations that preserve properties of zeros, such as all being real.

The link between polynomials with special properties and numerical integration is best known in Gauss quadrature: if w is a positive weight function on an interval $[a, b]$, then the Gauss quadrature rule asserts that

$$\int_a^b Pw = \sum_{j=1}^{n} w_j P\left(x_j\right), \tag{1.2}$$

for all polynomials P of degree $\leq 2n-1$. Here all $\left\{w_j\right\}$ are positive, while the quadrature points $\left\{x_j\right\}$ are distinct and lie in (a, b). Of course, they are the zeros of the orthogonal polynomial p_n of degree n, satisfying

$$\int_a^b p_n(x)\, x^j w(x)\, dx = 0,\ \ 0 \leq j \leq n-1. \tag{1.3}$$

Quadrature formulae (1.2) that integrate polynomials P of degree $\leq n-1$, are called interpolatory, while precisions in between $n-1$ and $2n-1$ have also been widely studied [8–11]. In all cases, the polynomial whose zeros are the quadrature points is a key tool.

The second author introduced quadrature formulae [12,13] that have their origins in convergence acceleration. The quadrature points are zeros of polynomials satisfying, for example,

$$\int_0^1 p_n(x)\,(\log x)^j\, dx = 0,\ \ j < n.$$

In Section 2 we shall survey some polynomials that are orthogonal to powers of a fixed function, such as $\log x$. In Section 3, we discuss polynomials that are orthogonal to exponentials or measures. In Section 4 we discuss positivity of the weights in interpolatory quadrature formulae generated by these polynomials. In Section 5, we discuss the use of potential theory to study asymptotics and zero distributions.

2. Polynomials orthogonal to powers of a fixed function

In investigating the T-transformation of Levin [14] (see also [12,13], [15, Chapter 19]) for accelerating convergence of sequences, and for related rational interpolation and interpolatory quadrature rules, the second author studied [16,17] what are now called the *Sidi polynomials*

$$D_n(z) = \sum_{j=0}^{n} \binom{n}{j} (j+1)^n (-z)^j. \tag{2.1}$$

They are uniquely determined, up to a multiplicative constant, by the orthogonality conditions

$$\int_0^1 D_n(x)\,(\log x)^j\, dx = 0,\ \ 0 \leq j \leq n-1. \tag{2.2}$$

Establishing the orthogonality relation from the definition (2.1) is straightforward. The orthogonality conditions imply that D_n has n distinct zeros in $(0, 1)$. Here are some further elementary properties, which can be proved by integration by parts, and Cauchy's integral formula:

Proposition 2.1. *(a) There is a Rodrigues type formula*

$$e^z D_n\left(e^z\right) = \left(\frac{d}{dz}\right)^n \left[e^z\left(1-e^z\right)^n\right].$$

(b) There is a contour integral representation

$$e^z D_n\left(e^z\right) = \frac{n!}{2\pi i}\int_C \frac{e^t}{t-z}\left(\frac{1-e^t}{t-z}\right)^n dt$$

where C is a simple closed curve encircling z.

The asymptotic behavior of these polynomials was investigated in [18], using the method of steepest descent. Let

$$\mathcal{A} = \left\{z = x+iy : x \geq 0, y \in (-\pi, \pi) \text{ and } 0 < |z-1|^2 < \left(\frac{y}{\sin y}\right)^2\right\}.$$

It is an unbounded doubly connected region inside the strip $|\mathrm{Im}\, z| < \pi$, $\mathrm{Re}\, z > 0$. Let

$$\Psi(z) = \frac{1}{e^z(1-z)}, z \in \overline{\mathcal{A}}.$$

It is shown in [18] that Ψ maps $\mathcal{A}$ conformally onto $\mathbb{C}\backslash[0,1]$. We let $\Phi = \Psi^{[-1]}$ denote the inverse conformal map of $\mathbb{C}\backslash[0,1]$ onto $\mathcal{A}$. We also need the function

$$h(b) = \frac{\sin b}{b} e^{b\cot b - 1}, b \in [0, \pi].$$

It was shown in [18] that h decreases from $h(0) = 1$ to $h(\pi) = 0$. Let $h^{[-1]} : [0,1] \to [0,\pi]$ denote the inverse function, and

$$g(x) = 1 - h^{[-1]}(x) \cot\left(h^{[-1]}(x)\right).$$

The asymptotics established there were

Theorem 2.2. *As $n \to \infty$,*

(a) Uniformly for z in compact subsets of $\mathbb{C}\backslash[0,1]$,

$$D_n(z) = \frac{n! e^{\Phi(z)}}{\sqrt{2\pi n \Phi(z)}} (-z\Phi(z))^n (1 + o(1)).$$

(b) Uniformly for x in compact subsets of $(0,1)$,

$$\begin{aligned} &D_n(x) \\ &= n! \left(\frac{2}{n\pi}\right)^{1/2} (-x e^{g(x)})^n \\ &\times \left\{ \frac{e^{g(x)}}{\left|g(x) + i h^{[-1]}(x)\right|^{1/2}} \cos\left[(n+1) h^{[-1]}(x) - \frac{1}{2}\arctan\left(\frac{h^{[-1]}(x)}{g(x)}\right)\right] + o(1) \right\}. \end{aligned}$$

The zero distribution of D_n also involves the inverse function of h : if

$$\nu_n = \frac{1}{n} \sum_{x: D_n(x)=0} \delta_x$$

is the zero counting measure of D_n, then it was proved that ν_n converges weakly to an absolutely continuous measure with derivative $-\frac{1}{\pi h'\left(h^{[-1]}(x)\right)}$ on $[0,1]$. The asymptotics in [18] were subsequently generalized by Elbert [19] and Zhao [20]. Surprisingly, the precise asymptotics in [18] were not sufficient to prove positivity of the weights in the interpolatory quadrature generated by D_n. This was only resolved in the affirmative much later, see Section 4.

We note that the second author also considered more general polynomials [16,17,21, pp. 845–6] given by

$$D_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (\beta + i + 1)^{\alpha+n} x^i, \tag{2.3}$$

where $\alpha, \beta > -1$. If α is a nonnegative integer, these admit the Rodrigues type representation

$$D_n^{(\alpha,\beta)}(x) = (-1)^n x^{-\beta-1} \left(x\frac{d}{dx}\right)^{\alpha+n} \left[x^{\beta+1}(1-x)^n\right]. \tag{2.4}$$

For general $\alpha, \beta > -1$, these polynomials satisfy the biorthogonality relation [17, p. 846], [21]

$$\int_0^1 D_n^{(\alpha,\beta)}(x) \left(\log x^{-1}\right)^{j+\alpha} x^\beta dx = 0, 0 \le j \le n-1. \tag{2.5}$$

A natural question is what happens if we replace $\log x$ in (2.2) by other functions. Herbert Stahl and the first author [22] investigated polynomials orthogonal to general powers of x. Closely related polynomials arise in the Borodin–Muttalib ensemble [23–26] but with varying exponential weights. Stahl and the first author proved [22, Theorem 1]

Theorem 2.3. *Let $\alpha > 0$ and*

$$S_n(x) = \sum_{j=0}^{n} \binom{n}{j} \left[\prod_{k=0}^{n-1} \left(k + \frac{j+1}{\alpha}\right)\right] (-x)^j.$$

Then

(a)

$$\int_0^1 S_n(x)\, x^{\alpha j} dx = 0,\, 0 \le j \le n-1. \tag{2.6}$$

(b) There is a Rodrigues type formula

$$S_n\left(x^{1/\alpha}\right) = x^{1-1/\alpha}\left(\frac{d}{dx}\right)^n \left[x^{n-1+1/\alpha}\left(1 - x^{1/\alpha}\right)\right]^n. \tag{2.7}$$

(c) There is a contour integral formula

$$S_n\left(z^{1/\alpha}\right) = \frac{n! z^{1-1/\alpha}}{2\pi i}\int_\Gamma \frac{t^{-1+1/\alpha}}{t-z}\left[\frac{t\left(1-t^{1/\alpha}\right)}{t-z}\right]^n dt.$$

Here $z \in \mathbb{C}\backslash(-\infty, 0]$ *while* Γ *is a simple closed contour in* $\mathbb{C}\backslash(-\infty, 0]$ *enclosing* z.

(d) There is confluence to the Sidi polynomials

$$\lim_{\alpha \to 0+} \alpha^n S_n(z) = D_n(z).$$

The above properties follow in a fairly straightforward way. There were also partial results about the zero distribution. The condition (2.6) ensures that the zeros of S_n are distinct and lie in $(0, 1)$. We applied results of VanAssche, Fano and Ortolani [27] to obtain asymptotics for ratios of coefficients in S_n, and hence to describe the limiting zero distribution of the reflected zeros, assuming that it exists:

Let

$$\nu_n = \frac{1}{n}\sum_{x: S_n(x)=0} \delta_{-x}.$$

so that ν_n is supported on $[-1, 0]$. Also let H denote the Hilbert transform, so that for functions $g \in L_1(\mathbb{R})$,

$$H[g](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{g(t)}{t-x} dt,$$

where PV denotes Cauchy principal value. We proved [22, Theorem 2]:

Theorem 2.4. *Let* $\alpha > 0$ *and*

$$f(x) = (1-x)^{1+1/\alpha} x^{-1} (\alpha + 1 - x)^{-1/\alpha},\, x \in (0, 1).$$

Then f *is strictly decreasing with inverse* $f^{[-1]}$. *Assume that the reflected zero counting measures* $\{\nu_n\}$ *converge weakly to some measure* ν *on* $[-1, 0]$. *Assume also that* ν *is absolutely continuous. Then*

$$\nu'(x) = -\frac{1}{\pi^2 x} H\left[f^{[-1]}\right](x),\, x \in (-1, 0).$$

It would obviously be preferable to have a more explicit form for the zero distribution, and a proof that the weak limit exists. In a subsequent paper, the first author and Soran [28] investigated polynomials satisfying a version of (2.6) with a weight. Recall that for ordinary orthogonal polynomials, the classical weights (Jacobi, Laguerre, Hermite) are characterized by their orthogonal polynomials admitting a Rodrigues formula. We characterized situations for these biorthogonal polynomials, where there is a Rodrigues formula:

Theorem 2.5. *Let* $\alpha > 0$. *Let* $w : (0, 1) \to (0, \infty)$ *be infinitely differentiable and positive a.e. on* $(0, 1)$. *Assume that* P_n *is a polynomial of degree* n *satisfying*

$$\int_0^1 P_n(x)\, x^{\alpha j} w(x)\, dx = 0,\, 0 \le j \le n-1. \tag{2.8}$$

Then P_n *admits a Rodrigues type formula*

$$P_n\left(x^{1/\alpha}\right) = \frac{x^{1-1/\alpha}}{w\left(x^{1/\alpha}\right)}\left(\frac{d}{dx}\right)^n \left\{x^{1/\alpha-1} w\left(x^{1/\alpha}\right)\left(x\left(1-x^{1/\alpha}\right)\right)^n\right\}$$

iff w *is a Jacobi weight*

$$w(x) = x^a (1-x)^b$$

for some $a, b > -1$.

There were also analogous results for the intervals $(0, \infty)$ and $(-\infty, \infty)$, as well as generating functions involving contour integrals. In a random matrix setting, Claeys and Wang [29] considered the case $\psi(x) = e^x$. Thus one forms polynomials orthogonal to $\left\{e^{kx}\right\}_{k=0}^{n-1}$. Using a Riemann–Hilbert formulation for the biorthogonal polynomials, and the Deift–Zhou steepest descent method, they obtained precise asymptotics for the polynomials.

3. Polynomials orthogonal to exponentials and measures

In [16] and [30] (see also [31]), the second author developed some new quadrature rules, with origins in techniques for accelerating convergence of sequences, for integrals of the form

$$I(f) = \int_a^b wf.$$

Here (a, b) can be a finite or infinite interval, and w is a suitable weight function. In particular, the following three cases were considered:

(i) $(a, b) = (0, 1)$ and $w(x) = x^\alpha (1-x)^\beta \left(\log x^{-1}\right)^\gamma$, where $\alpha > -1$ and $\beta + \gamma > -1$.
(ii) $(a, b) = (0, \infty)$ and $w(x) = x^\alpha e^{-x}$ with $\alpha > -1$ and $w(x) = x^\alpha E_p(x)$, with $p + \alpha > 0$, where $E_p(x) = \int_1^\infty e^{-xt} t^{-p} dt$ is the exponential integral.
(iiii) $(a, b) = (-\infty, \infty)$ and $w(x) = |x|^\beta e^{-x^2}$, $\beta > -1$.

The quadrature formulae take the form

$$I_n(f) = \sum_{j=1}^n w_{nj} f\left(x_{nj}\right)$$

and are chosen so that

$$H_n(z) = \sum_{j=1}^n \frac{w_{nj}}{z - x_{nj}}$$

approximates the Stieltjes transform

$$H(z) = \int_a^b \frac{w(x)}{z-x} dx,$$

for z outside (a, b). Starting with partial sums of the moment series

$$H(z) \sim \sum_{j=1}^\infty \frac{\mu_{j-1}}{z^j}, \ \mu_j = \int_a^b w(x) x^j dx$$

which may or may not converge, we apply a method of convergence acceleration. In particular, the Levin $\mathcal{L}$ and Sidi $\mathcal{S}$ transformation (see [14] and [15, Chapter 19]) yield good candidates.

This work was continued in [32] for the interval $(a, b) = (0, \infty)$ and $w(x) = x^\alpha e^{-x}$ and $w(x) = x^\alpha E_p(x)$. Applying the Levin transformation to an asymptotic expansion for $\int_0^\infty \frac{w(x)}{z-x} dx$, leads to the polynomials [30,33], [32, p. 214]

$$D_n^{[j]}(z) = (-1)^n \frac{n!}{\Gamma(\alpha + j + n + 1)} \frac{1}{z} \left(z \frac{d}{dz}\right)^n \left[z^{j+1} L_n^{(\alpha+j)}(z)\right], \tag{3.1}$$

where $L_n^{(\beta)}(z)$ is the classical Laguerre polynomial. The following was established in these works:

Proposition 3.1. *(a) $D_n^{[j]}$ satisfies the biorthogonality relations*

$$\int_0^\infty D_n^{[j]}(x) e^{-\sigma_{n,k} x} x^\alpha dx = 0, \ 1 \le k \le n, \tag{3.2}$$

where $\left\{\sigma_{n,k}\right\}$ are distinct and positive, and $\sigma_{n,k}^{-1}$ are positive roots of the polynomial

$$\psi_n(z) = (-1)^n z^{-j-1} \left(z \frac{d}{dz}\right)^n \left[z^{j+1} (1-z)^n\right].$$

(b) D_n has n simple positive roots $\left\{x_{nj}\right\}$ and a root at 0 of multiplicity j. This yields a quadrature rule

$$I_n^{[j]}[f] = \sum_{k=0}^{j-1} \bar{w}_k f^{(k)}(0) + \sum_{j=1}^n w_j f\left(x_{nj}\right), \tag{3.3}$$

that is exact, that is satisfies

$$I_n^{[j]}[f] = \int_0^\infty f(x) x^\alpha e^{-x} dx \tag{3.4}$$

for functions of the form

$$f(x) = P(x) + D_n^{[j]}(x) \sum_{k=1}^{n} c_k e^x e^{-\sigma_{n,k} x}, \tag{3.5}$$

where P is any polynomial of degree at most $j+n-1$, while the coefficients $\{c_k\}$ in the exponential part are arbitrary. In particular, for $j=0$, we obtain an interpolatory quadrature.

From the Sidi transformation, the authors obtained instead [32, p. 215] the polynomials

$$\hat{D}_n^{[j]}(z) = (-1)^n \frac{n!}{\Gamma(\alpha+j+n+1)} \left(\frac{d}{dz}\right)^n \left[z^{j+n} L_n^{(\alpha+j)}(z)\right] \tag{3.6}$$

and established the following:

Proposition 3.2. *(a) $\hat{D}_n^{[j]}$ satisfies the biorthogonality relations (3.2), where $\{\sigma_{n,k}\}$ are distinct and positive, and $\sigma_{n,k}^{-1}$ are positive roots of the polynomial*

$$\hat{\psi}_n(z) = (-1)^n z^{-j} \left(\frac{d}{dz}\right)^n \left[z^{j+n}(1-z)^n\right].$$

(b) $\hat{D}_n^{[j]}$ has n simple positive roots $\{x_{nj}\}$ and a root at 0 of multiplicity j. This yields a quadrature rule of the form

$$I_n^{[j]}[f] = \sum_{k=0}^{j-1} \bar{w}_k f^{(k)}(0) + \sum_{j=1}^{n} w_j f(x_{nj}), \tag{3.7}$$

that is exact, that is, satisfies (3.4) for functions of the form (3.5).

Related results for symmetric weights, such as $(1-x^2)^\alpha \left(\log\left(1-x^2\right)^{-1}\right)^\beta$ on $(-1,1)$, were considered by the second author in [34] .

In a 2008 paper [35], the authors considered polynomials P_n of degree n determined by biorthogonality conditions like (3.2), but in a more general setting:

Proposition 3.3. *Fix n distinct exponents $\{\sigma_{n,j}\}_{j=1}^n$ in $(0,\infty)$ and $\alpha > -1$. Determine P_n of degree n by the conditions*

$$\int_0^\infty P_n(x) e^{-\sigma_{n,j} x} x^\alpha dx = 0,\ 1 \le j \le n. \tag{3.8}$$

Define the associated exponential polynomial

$$Q_n(x) = \prod_{j=1}^{n} \left(x + \sigma_{n,j}^{-1}\right) = \sum_{j=0}^{n} q_{n,j} x^j. \tag{3.9}$$

Then

(a)

$$P_n(x) = \sum_{j=0}^{n} (-1)^{n-j} q_{n,j} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+j+1)} x^j.$$

(b)

$$Q_n(y) = \frac{(-1)^n}{\Gamma(\alpha+n+1)} \int_0^\infty e^{-t} t^\alpha P_n(-yt)\, dt.$$

(c)

$$x^\alpha P_n(x) = \frac{(-1)^n \Gamma(\alpha+n+1)}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{sx} s^{-\alpha-1} Q_n\left(-s^{-1}\right) ds,$$

where $\gamma > 0$ and the contour of integration is the line $\mathrm{Re}\, s = \gamma$.

Let

$$\nu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{-1/\sigma_{n,j}}$$

denote the zero counting measure for Q_n, and if $\{x_{n,j}\}$ are the zeros of P_n,

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{-x_{n,j}/(4n)}$$

so that μ_n is a contracted zero counting measure for P_n. The following result was proved there [35, p. 347, Thm. 1.2]:

Theorem 3.4. *Let $B > 0$. Assume that for $n \geq 1$, we are given distinct $\{\sigma_{n,j}\}_{j=1}^{n}$ in $[B, \infty)$. The following are equivalent:*

(a) There exists a measure ν such that $\nu_n \to \nu$ weakly as $n \to \infty$.
(b) There exists a measure μ such that $\mu_n \to \mu$ weakly as $n \to \infty$.

Moreover, assuming the weak convergence, both μ and ν have support in $\left[-\frac{1}{B}, 0\right]$, and μ will have a point mass at 0 of size λ iff ν does. Moreover, uniformly for $z \in \mathbb{C}\backslash\left[-\frac{1}{B}, 0\right]$,

$$\lim_{n\to\infty} |P_n(-4nz)|^{1/n}/(4n) = \exp\left(\int_{-1/B}^{0} \log|z-t|\, d\mu(t)\right).$$

The measures ν and μ were also related to functions defined by the asymptotic behavior of ratios or nth roots of the coefficients of P_n and Q_n, while various examples were presented there. As noted above, Claeys and Wang [29] investigated a class of polynomials that are biorthogonal to exponentials, with the added complication of an external field. These arise in a random matrix context, namely random matrices with equispaced external source. They obtained precise asymptotics using deep Riemann–Hilbert methods. Related themes have been considered, for example, in [26,36].

In a 2013 paper [37], the authors considered a more general orthogonality, to dilations of measures μ, supported on the real line, with all moments

$$\mu_j = \int_0^\infty x^j\, d\mu(x), j = 0, 1, 2, \ldots, \tag{3.10}$$

finite. We assume that for $n \geq 1$, we are given distinct positive numbers $\{\sigma_{n,j}\}_{j=1}^{n}$, and determine a monic polynomial P_n of degree n by the conditions

$$\int_0^\infty P_n(x)\, d\mu(\sigma_{n,j}x) = 0,\ 1 \leq j \leq n. \tag{3.11}$$

Equivalently,

$$\int_0^\infty P_n\left(\frac{t}{\sigma_{n,j}}\right) d\mu(t) = 0,\ 1 \leq j \leq n. \tag{3.12}$$

As in [35], P_n is closely related to the polynomial,

$$R_n(y) = \prod_{j=1}^{n}\left(y + \sigma_{n,j}^{-1}\right) = \sum_{j=0}^{n} r_{n,j} y^j. \tag{3.13}$$

which we called the *dilation polynomial associated with P_n*. The following simple proposition established the relationship between P_n and R_n:

Theorem 3.5. *Let μ be a positive measure on $(0, \infty)$ with infinitely many points in its support, and finite moments $\{\mu_j\}$. Let $\{\sigma_{n,j}\}_{j=1}^{n}$ be distinct positive numbers. Let P_n be a monic polynomial of degree n, determined by the orthogonality relations (3.11), and let R_n be given by (3.13). Then P_n exists, is unique, and*

(I)

$$P_n(x) = (-1)^n \sum_{j=0}^{n} r_{n,j}\frac{\mu_n}{\mu_j}(-x)^j, \tag{3.14}$$

while

$$(-1)^n R_n(-y) = \frac{1}{\mu_n}\int_0^\infty P_n(ty)\, d\mu(t). \tag{3.15}$$

(II) There exists $r > 0$ such that

$$P_n(x) = \mu_n \frac{(-1)^n}{2\pi i}\int_{|t|=r} R_n\left(-\frac{x}{t}\right) G(t)\frac{dt}{t} \tag{3.16}$$

where

$$G(t)=\sum_{j=0}^{\infty}\frac{t^j}{\mu_j}. \tag{3.17}$$

(III) Write $\sigma_j=\sigma_{n,j}$, $1\le j\le n$. *Then*

$$\frac{P_n(x)}{\mu_n}=\frac{\det\begin{bmatrix}1 & \sigma_1^{-1} & \sigma_1^{-2} & \cdots & \sigma_1^{-n}\\ 1 & \sigma_2^{-1} & \sigma_2^{-2} & \cdots & \sigma_2^{-n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \sigma_n^{-1} & \sigma_n^{-2} & \cdots & \sigma_n^{-n}\\ \frac{1}{\mu_0} & \frac{x}{\mu_1} & \frac{x^2}{\mu_2} & \cdots & \frac{x^n}{\mu_n}\end{bmatrix}}{\det\begin{bmatrix}1 & \sigma_1^{-1} & \sigma_1^{-2} & \cdots & \sigma_1^{-n+1}\\ 1 & \sigma_2^{-1} & \sigma_2^{-2} & \cdots & \sigma_2^{-n+1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \sigma_n^{-1} & \sigma_n^{-2} & \cdots & \sigma_n^{-n+1}\end{bmatrix}}. \tag{3.18}$$

(IV) If μ *has form*

$$d\mu(t)=t^{\alpha}e^{-t^{\beta}}dt,\ t\in(0,\infty),$$

where $\alpha>-1$, $\beta>0$, *then* P_n *has* n *simple zeros in* $(0,\infty)$.

Parts of Theorem 3.4 overlap with results of Brezinski [1], Iserles, Nørsett and Saff [6,7] on biorthogonal polynomials in a more general setting. In a special case, we gave a simple new contour integral representation of P_n :

Theorem 3.6. *Let* $\beta\ge 1$, $\alpha>-1$, *and* $d\mu(t)=t^{\alpha}e^{-t^{\beta}}dt$, $t\in(0,\infty)$. *Let* $\{\sigma_{n,j}\}_{j=1}^{n}$ *be distinct positive numbers. Let* P_n *be a monic polynomial of degree* n , *determined by the orthogonality relations (3.11), and* R_n *be given by (3.13). Let*

$$\frac{\pi}{2\beta}<\eta<\frac{\pi}{\beta}, \tag{3.19}$$

$s>0$, *and let* Γ *be the contour consisting of the rays* $\Gamma_+=\{re^{i\eta}:\ r\ge s\}$, $\Gamma_-=\{re^{-i\eta}:\ r\ge s\}$, *and the circular arc* $\Gamma_s=\{se^{i\theta}:|\theta|\le\eta\}$. *Assume that* Γ *is traversed in such a way that* Γ_s *is traversed anticlockwise. Then for all complex* z,

$$P_n(z)=\frac{\beta^2(-1)^n\mu_n}{2\pi i}\int_{\Gamma}e^{t^{\beta}}t^{\beta-\alpha-2}R_n\left(-\frac{z}{t}\right)dt. \tag{3.20}$$

Using this and standard techniques for asymptotics of contour integrals, we showed in [37] that "strong" asymptotics for R_n lead to strong asymptotics for P_n.

4. Positive quadrature rules generated by biorthogonal polynomials

One of the most important questions about any quadrature formula is the positivity of its weights. For interpolatory formulae that are not Gauss quadratures, this is often difficult to establish. The setting is as follows: given an interval $[a,b]$ and $a\le x_1<x_2<\cdots<x_n\le b$, a weight function w, we determine $\{\lambda_j\}_{j=1}^{n}$ so that

$$\sum_{j=1}^{n}\lambda_jP(x_j)=\int_a^n Pw.$$

Are the $\{\lambda_j\}$ positive?

For the quadrature determined by the Sidi polynomials, this question had been open since 1980. It was resolved affirmatively by the authors in a 2010 paper [21]. We considered a continuously differentiable, strictly increasing, function $\varphi:(a,b)\to\mathbb{R}$, a positive weight function w, and the monic polynomial p_n of degree n determined by the conditions

$$\int_a^b p_n\varphi^j w=0,\ 0\le j\le n-1. \tag{4.1}$$

Let $\{x_j\}_{j=1}^{n}$ denote the zeros of p_n in (a,b). The corresponding interpolatory quadrature is exact for polynomials P of degree $<n$:

$$\sum_{j=1}^{n}\lambda_jP(x_j)=\int_a^b Pw. \tag{4.2}$$

Recall that a function g is said to be m *absolutely monotone* in an interval J if $g^{(m)}$ exists there and

$$g^{(j)} > 0 \text{ in } J \text{ for } 0 \le j \le m.$$

If

$$(-1)^j g^{(j)} > 0 \text{ in } J \text{ for } 0 \le j \le m,$$

g is said to be m *completely monotone* in J. Our main result was:

Theorem 4.1. *Let $n \ge 1$ and $\varphi : (a, b) \to \mathbb{R}$ be a strictly increasing function with $n-1$ continuous derivatives, and let ψ denote its inverse function, with domain of definition $I = \{\varphi(x) : x \in (a, b)\}$. Assume that for each $\beta \in I$, the function*

$$g(t) = \frac{1}{\psi(\beta) - \psi(t)}, \quad t \in I \backslash \{\beta\}, \tag{4.3}$$

is $n-1$ absolutely monotone in $I \cap (-\infty, \beta)$ and $-g$ is $n-1$ completely monotone in $I \cap (\beta, \infty)$. Let $w : (a, b) \to (0, \infty)$ be such that $\int_a^b x^j \varphi(x)^k w(x)\, dx$ is defined and finite for $0 \le j \le n$ and $0 \le k \le n-1$. Let p_n be the monic polynomial of degree n determined by the biorthogonality conditions (4.1). Then the weights $\{\lambda_j\}_{j=1}^n$ in the interpolatory rule I_n generated by p_n and w are all positive.

We also proved positivity of the quadrature weights when the weight w is replaced by $w\,|\varphi|$:

Theorem 4.2. *Assume the hypotheses of Theorem 4.1, and in addition, that φ is of one sign in (a, b). Then the weights $\{\lambda_j\}_{j=1}^n$ in the interpolatory rule I_n generated by p_n and $\hat{w} = w\,|\varphi|$ are all positive.*

Corollary 4.3. *Let $\alpha, \beta > -1$ and $n \ge 1$. Let $w(x) = \left(\log x^{-1}\right)^{\alpha} x^{\beta}$ or $w(x) = \left(\log x^{-1}\right)^{\alpha+1} x^{\beta}$, $x \in (0, 1)$. Then the weights $\{\lambda_j\}_{j=1}^n$ in the interpolatory rule generated by the Sidi polynomials $D_n^{(\alpha,\beta)}$ defined by (2.3) and the weight w are positive.*

We also considered the case where the quadrature points come from a different weight. The proofs of these results are non-trivial. They involve careful zero counting arguments.

Here is an interesting unsolved problem:

Problem 4.4. Investigate the positivity of interpolatory quadrature generated by the polynomials defined by (3.1) and (3.6), that are biorthogonal to exponentials.

The authors spent much effort trying to use the techniques of [21] for this problem, but failed.

5. Potential theory

Potential theory is a powerful tool in so many problems involving polynomials [38–41]. In this section, we discuss its application to biorthogonal polynomials. Let $\mathcal{P}(\mathcal{K})$ denote the set of all probability measures with compact support contained in the set $\mathcal{K}$. For any positive Borel measure μ, we define its energy integral

$$\mathcal{I}(\mu) = \iint \log \frac{1}{|x-t|} d\mu(x)\, d\mu(t). \tag{5.1}$$

For $\mathcal{K} \subset \mathbb{C}$, its (inner) logarithmic capacity is

$$cap(\mathcal{K}) = \sup\left\{e^{-\mathcal{I}(\mu)} : \mu \in \mathcal{P}(\mathcal{K})\right\}.$$

We say that a property holds q.e. (quasi-everywhere) if it holds outside a set of capacity 0. We use *meas* to denote linear Lebesgue measure. For further orientation, see for example [39,40,42].

Let $\psi : [0, 1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function, with inverse $\psi^{[-1]}$, and determine a monic polynomial P_n of degree n by the biorthogonality conditions

$$\int_0^1 P_n(x)\, \psi(x)^j\, dx = \begin{cases} 0, & j = 0, 1, 2, \dots, n-1, \\ I_n \neq 0, & j = n \end{cases}. \tag{5.2}$$

P_n will have n simple zeros in $(0, 1)$, so we may write

$$P_n(x) = \prod_{j=1}^{n} \left(x - x_{jn}\right). \tag{5.3}$$

Define the zero counting measures

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{jn}}. \tag{5.4}$$

We need a new energy integral

$$\mathcal{J}(\mu)=\iint K(x,t)\,d\mu(x)\,d\mu(t), \tag{5.5}$$

and a new kernel

$$K(x,t)=\log\frac{1}{|x-t|}+\log\frac{1}{|\psi(x)-\psi(t)|}. \tag{5.6}$$

The minimal energy corresponding to ψ is

$$\mathcal{J}^*(\psi)=\inf\{\mathcal{J}(\mu):\mu\in\mathcal{P}([0,1])\}. \tag{5.7}$$

For probability measures μ,ν, we define the classical potential

$$U^{\mu}(x)=\int\log\frac{1}{|x-t|}d\mu(t), \tag{5.8}$$

the mixed potential

$$W^{\mu,\nu}(x)=\int\log\frac{1}{|x-t|}d\mu(t)+\int\log\frac{1}{|\psi(x)-\psi(t)|}d\nu(t) \tag{5.9}$$

$$=U^{\mu}(x)+U^{\nu\circ\psi^{[-1]}}\circ\psi(x), \tag{5.10}$$

and the ψ potential

$$W^{\mu}(x)=W^{\mu,\mu}(x)=\int K(x,t)\,d\mu(t). \tag{5.11}$$

We needed ψ to map small sets to small sets:

Definition 5.1. Let $\psi:[0,1]\to[\psi(0),\psi(1)]$ be a strictly increasing continuous function, with inverse $\psi^{[-1]}$. Assume that ψ satisfies the following two conditions:

(I)

$$cap(E)=0\Rightarrow cap\left(\psi^{[-1]}(E)\right)=0. \tag{5.12}$$

(II) For each $\varepsilon>0$, there exists $\delta>0$ such that

$$meas(E)\le\delta\Rightarrow meas\left(\psi^{[-1]}(E)\right)\le\varepsilon. \tag{5.13}$$

Then we say that ψ preserves smallness of sets.

The conditions (I), (II) are satisfied if ψ satisfies a local lower Lipschitz condition. Using classical methods, we proved in [43, p. 29, Thm. 1.2]:

Theorem 5.2. *Let $\psi:[0,1]\to[\psi(0),\psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. Define the minimal energy $\mathcal{J}^*=\mathcal{J}^*(\psi)$ by (5.7). Then*

(a) $\mathcal{J}^*$ is finite and there exists a unique probability measure ν_ψ on $[0,1]$ such that

$$\mathcal{J}\left(\nu_\psi\right)=\mathcal{J}^*. \tag{5.14}$$

(b)

$$W^{\nu_\psi}\ge\mathcal{J}^*\text{ q.e. in }[0,1]. \tag{5.15}$$

In particular, this is true at each point of continuity of W^{ν_ψ}.

(c)

$$W^{\nu_\psi}\le\mathcal{J}^*\text{ in }supp\left[\nu_\psi\right]. \tag{5.16}$$

and

$$W^{\nu_\psi}=\mathcal{J}^*\text{ q.e. in }supp\left[\nu_\psi\right]. \tag{5.17}$$

(d) ν_ψ is absolutely continuous with respect to linear Lebesgue measure on $[0,1]$. Moreover, there are constants C_1 and C_2 depending only on ψ, such that for all compact $\mathcal{K}\subset[0,1]$,

$$\nu_\psi(K)\le\frac{C_1}{|\log cap(\mathcal{K})|}\le\frac{C_2}{|\log meas(\mathcal{K})|}. \tag{5.18}$$

(e) There exists $\varepsilon>0$ such that

$$[0,\varepsilon]\cup[1-\varepsilon,1]\subset supp\left[\nu_\psi\right]. \tag{5.19}$$

Let

$$I_n = \int_0^1 P_n(t)\, \psi(t)^n\, dt,\ \ n \geq 1. \tag{5.20}$$

Theorem 5.3. *Let $\psi : [0, 1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. Let $\{P_n\}$ be the corresponding biorthogonal polynomials, with zero counting measures $\{\mu_n\}$. If*

$$supp\left[\nu_\psi\right] = [0, 1], \tag{5.21}$$

then the zero counting measures $\{\mu_n\}$ of (P_n) satisfy

$$\mu_n \overset{*}{\to} \nu_\psi, n \to \infty \tag{5.22}$$

and

$$\lim_{n\to\infty} I_n^{1/n} = \exp\left(-\mathcal{J}^*\right). \tag{5.23}$$

Moreover, uniformly for z in compact subsets of $\mathbb{C}\backslash[0, 1]$

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \exp\left(-U^\mu(z)\right). \tag{5.24}$$

We also proved that we can replace (5.21) by the more implicit, but more general, assumption that $\mathrm{supp}\left[\nu_\psi\right]$ contains the support of every weak limit of every subsequence of $\{\mu_n\}$. We proved (5.21) when the kernel K, and hence the potential W^{ν_ψ}, satisfies a convexity condition:

Theorem 5.4. *Let $\psi : [0, 1] \to [\psi(0), \psi(1)]$ be a strictly increasing continuous function that preserves smallness of sets. In addition assume that ψ is twice continuously differentiable in $(0, 1)$ and either*

(a) for $x, t \in (0, 1)$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2} K(x, t) > 0, \tag{5.25}$$

or

(b) for $x, t \in (\psi(0), \psi(1))$ with $x \neq t$,

$$\frac{\partial^2}{\partial x^2}\left[K\left(\psi^{[-1]}(x), \psi^{[-1]}(t)\right)\right] > 0. \tag{5.26}$$

Then

$$supp\left[\nu_\psi\right] = [0, 1]. \tag{5.27}$$

We showed in [43] that for

$$\psi(x) = x^\alpha, x \in [0, 1].$$

either (5.25) or (5.26) holds and hence (5.21) holds. We showed this separately for $\alpha \geq 1$ and for $\alpha < 1$. As noted before, such a ψ arises in the Borodin–Muttalib ensemble in random matrices [25]. Properties of the equilibrium measure were also investigated there.

Claeys and Wang [29] provided a detailed study for the case where $\psi(t) = e^t$, with the added complication of an external field. They established an explicit formula for the density of the equilibrium density ν_ψ in terms of the second derivative of the external field.

We believe the following problem is interesting:

Problem 5.5. Find general hypotheses for $\mathrm{supp}\left[\nu_\psi\right] = [0, 1]$.

Problem 5.6. Find classes of ψ for which we can explicitly solve the integral equation (5.17).

Bloom, Levenberg, Totik, and Wielonsky [44] considered a much more general setting where there is an external field, while ψ need not be strictly increasing. Let $\mathcal{K} \subset \mathbb{C}$ be closed and have positive capacity. Let $Q : \mathcal{K} \to \mathbb{R}$ be lower semicontinuous on $\mathcal{K}$, with

$$\lim_{|z|\to\infty, z\in\mathcal{K}} [Q(z) - \log|z|] = \infty.$$

Of course if $\mathcal{K}$ is compact, this last assumption is vacuous. For a given continuous function $\psi : \mathcal{K} \to \mathbb{C}$, we say Q is *ψ-admissible* if

$$\lim_{|z|\to\infty, z\in\mathcal{K}} [Q(z) - \log|z| - \log(1 + |\psi(z)|)] = \infty.$$

For probability measures μ on $\mathcal{K}$, define the energy integral with external field Q,

$$\mathcal{E}(\mu) = \iint K(x, y)\, d\mu(x)\ d\mu(y) + 2\int Q\ d\mu.$$

Here the kernel K is as in (5.6). We also need a "push forward" measure $\psi_*\mu$ defined by

$$\int h \circ \psi\ d\mu = \int h\, d\psi_*\mu,$$

so that in particular,

$$\iint \log \frac{1}{|\psi(x) - \psi(t)|} d\mu(x)\ d\mu(t) = \iint \log \frac{1}{|x - t|} d\psi_*\mu(x)\ d\psi_*\mu(t).$$

Let

$$V = \inf\{\mathcal{E}(\mu) : \mu \text{ is a probability measure on } \mathcal{K}\}.$$

Bloom, Levenberg, Totik, and Wielonsky proved that there is a unique minimizing measure:

Proposition 5.7. *Let $\mathcal{K} \subset \mathbb{C}$ be closed, and Q be ψ-admissible for $\mathcal{K}$. Suppose there exists a probability measure μ on $\mathcal{K}$ with $\mathcal{E}(\mu) < \infty$. Then*

(a) There is a unique probability measure μ_Q on $\mathcal{K}$ with

$$\mathcal{E}(\mu_Q) = V.$$

(b) μ_Q has compact support and $\mathcal{I}(\mu_Q)$ and $\mathcal{I}(\psi_*\mu_Q)$ are finite.

(c) The following Frostman type inequalities hold:

$$U^{\mu_Q}(z) + U^{\psi_*\mu_Q}(\psi(z)) + Q(z) \geq F \text{ q.e. on } \mathcal{K};$$
$$U^{\mu_Q}(z) + U^{\psi_*\mu_Q}(\psi(z)) + Q(z) \leq F \text{ on supp}\left[\mu_Q\right].$$

Here

$$F = V - \int Q\ d\mu_Q.$$

(d) If a probability measure μ on $\mathcal{K}$ has finite energy $\mathcal{E}(\mu)$ and satisfies

$$U^{\mu}(z) + U^{\psi_*\mu}(\psi(z)) + Q(z) \geq C \text{ q.e. on } \mathcal{K};$$
$$U^{\mu}(z) + U^{\psi_*\mu}(\psi(z)) + Q(z) \leq C \text{ on supp}\left[\mu_Q\right].$$

for some constant C then $\mu = \mu_Q$.

Bloom et al. also studied weighted Fekete points and convergence of their counting measures to the equilibrium measure. They establish inequalities that estimate the growth of $P(z)\,Q(\psi(z))$, where P and Q are polynomials. These are impressive extensions of the classical Bernstein–Walsh inequalities for growth of polynomials.

One problem that stands out both in this more general situation and the more restrictive situation in [43] is that not much is known about the support of the equilibrium measure μ_Q as well as the behavior of μ_Q. It certainly merits further investigation.

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