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# Exponential convergence of some recent numerical quadrature methods for Hadamard finite parts of singular integrals of periodic analytic functions

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## Abstract

Let

$$I[f] = \;=\!\!\!\!\!\!\int_a^b f(x)\,dx, \quad f(x) = \frac{g(x)}{(x-t)^m}, \quad m = 1, 2, \ldots, \quad a < t < b,$$

assuming that $g \in C^\infty[a,b]$ such that $f(x)$ is $T$-periodic, $T = b - a$, and $f(x) \in C^\infty(\mathbb{R}_t)$ with $\mathbb{R}_t = \mathbb{R}\backslash\{t + kT\}_{k=-\infty}^{\infty}$. Here $=\!\!\!\!\!\!\int_a^b f(x)\,dx$ stands for the Hadamard Finite Part (HFP) of the singular integral $\int_a^b f(x)\,dx$ that does not exist in the regular sense. In a recent work, we unified the treatment of these HFP integrals by using a generalization of the Euler–Maclaurin expansion due to the author and developed a number of numerical quadrature formulas $\widehat{T}^{(s)}_{m,n}[f]$ of trapezoidal type for $I[f]$ for all $m$. For example, three numerical quadrature formulas of trapezoidal type result from this approach for the case $m = 3$, and these are

$$\widehat{T}^{(0)}_{3,n}[f] = h\sum_{j=1}^{n-1} f(t+jh) - \frac{\pi^2}{3}\,g'(t)\,h^{-1} + \frac{1}{6}\,g'''(t)\,h, \quad h = \frac{T}{n},$$

$$\widehat{T}^{(1)}_{3,n}[f] = h\sum_{j=1}^{n} f(t+jh-h/2) - \pi^2\,g'(t)\,h^{-1}, \quad h = \frac{T}{n},$$

$$\widehat{T}^{(2)}_{3,n}[f] = 2h\sum_{j=1}^{n} f(t+jh-h/2) - \frac{h}{2}\sum_{j=1}^{2n} f(t+jh/2-h/4), \quad h = \frac{T}{n}.$$

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For all $m$ and $s$, we showed that all of the numerical quadrature formulas $\widehat{T}^{(s)}_{m,n}[f]$ have spectral accuracy; that is,

$$\widehat{T}^{(s)}_{m,n}[f] - I[f] = o(n^{-\mu}) \quad \text{as}\, n \to \infty \quad \forall \mu > 0.$$

In this work, we continue our study of convergence and extend it to functions $f(x)$ that possess certain analyticity properties. Specifically, we assume that $f(z)$, as a function of the complex variable $z$, is also analytic in the infinite strip $|\mathrm{Im}\, z| < \sigma$ for some $\sigma > 0$, excluding the poles of order $m$ at the points $t + kT$, $k = 0, \pm 1, \pm 2, \ldots$. For $m = 1, 2, 3, 4$ and relevant $s$, we prove that

$$\widehat{T}^{(s)}_{m,n}[f] - I[f] = O\big(\exp(-2\pi n\rho/T)\big) \quad \text{as}\, n \to \infty \quad \forall \rho < \sigma.$$

**Keywords** Hadamard finite part · Cauchy Principal Value · Singular integrals · Hypersingular integrals · Supersingular integrals · Numerical quadrature · Trapezoidal-like rules

**Mathematics Subject Classification** 41A55 · 41A60 · 45B05 · 45E05 · 65B15 · 65D30 · 65D32

## 1 Introduction and background

In a recent work [10], we considered the efficient numerical computation of

$$I[f] = \int_a^b f(x)\,dx, \quad f(x) = \frac{g(x)}{(x-t)^m}, \quad g \in C^\infty[a,b], \quad m = 1, 2, \ldots, \quad a < t < b, \tag{1.1}$$

such that

$$f(x) \text{ is } T\text{-periodic}, \quad f \in C^\infty(\mathbb{R}_t), \quad \mathbb{R}_t = \mathbb{R}\backslash\{t + kT\}_{k=-\infty}^{\infty}, \quad T = b - a. \tag{1.2}$$

Clearly, the integrals $\int_a^b f(x)\,dx$ are *not* defined in the regular sense, but they *are* defined in the sense of *Hadamard Finite Part (HFP)*, the HFP of $\int_a^b f(x)\,dx$ being commonly denoted by $⨎_a^b f(x)\,dx$.[1]

By invoking a recent generalization of the Euler–Maclaurin (E–M) expansion developed in Sidi [8, Theorem 2.3] that also applies to both regular and HFP integrals, we unified the treatments of the HFP integrals defined in (1.1)–(1.2) to cover all $m$. We thus derived a number of very effective numerical quadrature formulas $\widehat{T}^{(s)}_{m,n}[f]$,

[1] When $m = 1$, the HFP of $\int_a^b f(x)\,dx$ is also called its *Cauchy Principal Value (CPV)* and the accepted notation for it is $⨍_a^b f(x)\,dx$. When $m = 2$, $⨎_a^b f(x)\,dx$ is called a *hypersingular integral,* and when $m = 3$, $⨎_a^b f(x)\,dx$ is called a *supersingular integral.*
We reserve the notation $\int_a^b u(x)\,dx$ for integrals that exist in the regular sense.

$s = 0, 1, \ldots, \lceil (m+1)/2 \rceil$, for $I[f]$ and all $m \geq 1$.[2] (Recall that $\lceil a \rceil$ stands for the smallest integer greater than or equal to $a$.)

With $h = T/n$, $n = 1, 2, \ldots,$ and depending on whether $m$ is even or odd, the formulas $\widehat{T}^{(0)}_{m,n}[f]$, the most basic of the formulas $\widehat{T}^{(s)}_{m,n}[f]$, are obtained directly from the generalized E–M expansion and read as follows:

1. For $m = 2r$, $r = 1, 2, \ldots$:

$$\widehat{T}^{(0)}_{2r,n}[f] = h \sum_{j=1}^{n-1} f(t + jh) - 2 \sum_{i=0}^{r} \frac{g^{(2i)}(t)}{(2i)!} \zeta(2r - 2i) h^{-2r+2i+1}. \tag{1.3}$$

2. For $m = 2r + 1$, $r = 0, 1, 2, \ldots$:

$$\widehat{T}^{(0)}_{2r+1,n}[f] = h \sum_{j=1}^{n-1} f(t + jh) - 2 \sum_{i=0}^{r} \frac{g^{(2i+1)}(t)}{(2i+1)!} \zeta(2r - 2i) h^{-2r+2i+1}. \tag{1.4}$$

Here $\zeta(z)$ is the Riemann Zeta function.

Upon invoking the known fact that

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}, \quad k = 0, 1, \ldots; \quad B_s \text{ Bernoulli numbers},$$

(1.3) and (1.4) can be expressed more revealingly as

$$\widehat{T}^{(0)}_{2r,n}[f] = h \sum_{j=1}^{n-1} f(t + jh) + \sum_{i=0}^{r} (-1)^i \frac{g^{(2r-2i)}(t)}{(2r-2i)!} \frac{(2\pi)^i B_{2i}}{(2i)!} h^{-2i+1}, \tag{1.5}$$

and

$$\widehat{T}^{(0)}_{2r+1,n}[f] = h \sum_{j=1}^{n-1} f(t + jh) + \sum_{i=0}^{r} (-1)^i \frac{g^{(2r-2i+1)}(t)}{(2r-2i+1)!} \frac{(2\pi)^i B_{2i}}{(2i)!} h^{-2i+1}. \tag{1.6}$$

Note that the summations $\sum_{i=0}^{r}$ in (1.5) and (1.6) are linear combinations of $g^{(m)}(t)h,\ g^{(m-2)}(t)h^{-1},\ g^{(m-4)}(t)h^{-3}, \ldots\ g^{(m-2r)}(t)h^{1-2r}$, and this requires the computation of several derivatives of $g(x)$, which may be inconvenient or difficult in certain cases. This prompts us to eliminate some or all of these derivatives, starting with the highest order derivative $g^{(m)}(t)$, to obtain formulas less dependent on (or even independent of) the derivatives of $g(x)$. This amounts to eliminating the powers of $h$ from these summations in the order $h^1, h^{-1}, h^{-3}, \ldots, h^{1-2r}$, which we can

[2] A similar, yet somewhat different, treatment for the cases $m = 1$ and $m = 2$ was given earlier in Sidi [9]. The treatment of [10] was recently extended by the author in [12] to deal with nonperiodic HFP integrals $\oint_a^b g(x)/(x-t)^m\, dx$, $m = 1, 2, \ldots,$ where $g(x)$ is allowed to have arbitrary integrable singularities at the endpoints.

accomplish by a process that resembles the Richardson extrapolation in the reverse direction. (For Richardson extrapolation, see Sidi [6, Chapter 1], for example.) For each $s \geq 1$, we obtain $\widehat{T}^{(s)}_{m,n}[f]$ as a linear combination of $\widehat{T}^{(0)}_{m,n}[f]$, $\widehat{T}^{(0)}_{m,2n}[f]$, $\widehat{T}^{(0)}_{m,4n}[f]$, ..., $\widehat{T}^{(0)}_{m,2^s n}[f]$, by eliminating the terms involving the $s$ powers $h^1, h^{-1}, h^{-3}, \dots, h^{3-2s}$, in this order. Thus, $\widehat{T}^{(s)}_{m,n}[f]$ are all of the form

$$\widehat{T}^{(s)}_{m,n}[f] = \sum_{k=0}^{s} \alpha_k^{(s)} \widehat{T}^{(0)}_{m,2^k n}[f], \quad \sum_{k=0}^{s} \alpha_k^{(s)} = 1; \quad \alpha_k^{(s)} \text{independent of } m, n. \tag{1.7}$$

With $s = 1, 2, 3,$ for example, we have

$$\widehat{T}^{(1)}_{m,n}[f] = -\widehat{T}^{(0)}_{m,n}[f] + 2\widehat{T}^{(0)}_{m,2n}[f] \tag{1.8a}$$

$$\widehat{T}^{(2)}_{m,n}[f] = -2\widehat{T}^{(0)}_{m,n}[f] + 5\widehat{T}^{(0)}_{m,2n}[f] - 2\widehat{T}^{(0)}_{m,4n}[f] \tag{1.8b}$$

$$\widehat{T}^{(3)}_{m,n}[f] = -\frac{16}{7}\widehat{T}^{(0)}_{m,n}[f] + 6\widehat{T}^{(0)}_{m,2n}[f] - 3\widehat{T}^{(0)}_{m,4n}[f] + \frac{2}{7}\widehat{T}^{(0)}_{m,8n}[f] \tag{1.8c}$$

For $m = 1, 2, 3, 4,$ this procedure results in the following quadrature formulas.

1. *The case* m=1:

$$\widehat{T}^{(0)}_{1,n}[f] = h\sum_{j=1}^{n-1} f(t + jh) + g'(t)h \tag{1.9a}$$

$$\widehat{T}^{(1)}_{1,n}[f] = h\sum_{j=1}^{n} f(t + jh - h/2) \tag{1.9b}$$

2. *The case* m=2:

$$\widehat{T}^{(0)}_{2,n}[f] = h\sum_{j=1}^{n-1} f(t + jh) - \frac{\pi^2}{3}g(t)h^{-1} + \frac{1}{2}g''(t)h \tag{1.10a}$$

$$\widehat{T}^{(1)}_{2,n}[f] = h\sum_{j=1}^{n} f(t + jh - h/2) - \pi^2 g(t)h^{-1} \tag{1.10b}$$

$$\widehat{T}^{(2)}_{2,n}[f] = 2h\sum_{j=1}^{n} f(t + jh - h/2) - \frac{h}{2}\sum_{j=1}^{2n} f(t + jh/2 - h/4) \tag{1.10c}$$

3. *The case* m=3:

$$\widehat{T}^{(0)}_{3,n}[f] = h\sum_{j=1}^{n-1} f(t+jh) - \frac{\pi^2}{3}g'(t)h^{-1} + \frac{1}{6}g'''(t)h \tag{1.11a}$$

$$\widehat{T}^{(1)}_{3,n}[f] = h\sum_{j=1}^{n} f(t+jh-h/2) - \pi^2 g'(t)h^{-1} \tag{1.11b}$$

$$\widehat{T}^{(2)}_{3,n}[f] = 2h\sum_{j=1}^{n} f(t+jh-h/2) - \frac{h}{2}\sum_{j=1}^{2n} f(t+jh/2-h/4) \tag{1.11c}$$

4. *The case* m=4:

$$\widehat{T}^{(0)}_{4,n}[f] = h\sum_{j=1}^{n-1} f(t+jh) - \frac{\pi^4}{45}g(t)h^{-3} - \frac{\pi^2}{6}g''(t)h^{-1} + \frac{1}{24}g^{(4)}(t)h \tag{1.12a}$$

$$\widehat{T}^{(1)}_{4,n}[f] = h\sum_{j=1}^{n} f(t+jh-h/2) - \frac{\pi^4}{3}g(t)h^{-3} - \frac{\pi^2}{2}g''(t)h^{-1} \tag{1.12b}$$

$$\widehat{T}^{(2)}_{4,n}[f] = 2h\sum_{j=1}^{n} f(t+jh-h/2) - \frac{h}{2}\sum_{j=1}^{2n} f(t+jh/2-h/4) + 2\pi^4 g(t)h^{-3} \tag{1.12c}$$

$$\widehat{T}^{(3)}_{4,n}[f] = \frac{16h}{7}\sum_{j=1}^{n} f(t+jh-h/2) - \frac{5h}{7}\sum_{j=1}^{2n} f(t+jh/2-h/4) + \frac{h}{28}\sum_{j=1}^{4n} f(t+jh/4-h/8) \tag{1.12d}$$

In the process of derivation of our methods, we also proved in [10, Theorem 4.1] that *all* the quadrature formulas $\widehat{T}^{(s)}_{m,n}[f]$ have *spectral* convergence. We reproduce this theorem here for convenience. (For more on the case $m = 3$, see also [11].)

**Theorem 1.1** *Let $f(x)$ be as in (1.1)–(1.2), and let the numerical quadrature formulas $\widehat{T}^{(s)}_{m,n}[f]$ be as in (1.7). Then $\lim_{n\to\infty} \widehat{T}^{(s)}_{m,n}[f] = I[f]$, and we have*

$$\widehat{T}^{(s)}_{m,n}[f] - I[f] = o(n^{-\mu}) \quad \text{as}\, n\to\infty \quad \forall\mu > 0. \tag{1.13}$$

*In words, the errors in the $\widehat{T}^{(s)}_{m,n}[f]$ tend to zero as $n\to\infty$ faster than* every *negative power of $n$.*

We note that all the quadrature formulas $\widehat{T}^{(s)}_{m,n}[f]$ possess the following favorable properties:

1. Unlike most quadrature formulas in the literature, they are *compact* in that they consist of trapezoidal-like rules with simple, yet sophisticated and unexpected, “correction” terms to account for the singularity at $x = t$.
2. They have a unified convergence theory that follows directly from the way they are derived.
3. Unlike most quadrature formulas in the literature, which attain limited accuracies, our quadrature formulas enjoy *spectral* accuracy.
4. Because they enjoy spectral accuracy, when applied in floating-point arithmetic, they are much more stable numerically than existing methods.

For all the above, we refer the reader to [10].

In this work, we expand the convergence theory of Theorem 1.1, for $m = 1, 2, 3, 4$, to the cases in which the function $f(x)$, in addition to being as in (1.1)–(1.2), has an analytic continuation $f(z)$ as a function of the complex variable $z$ for $|\text{Im}\, z| < \sigma$, for some $\sigma > 0$. Thus, the main result of this paper is the following theorem that is much stronger than Theorem 1.1 under the given conditions:

**Theorem 1.2** *Let $f(x)$ be as in* (1.1)–(1.2). *Assume also that, as a function of the complex variable $z$, $f(z)$ is $T$-periodic and analytic in the infinite strip $D_\sigma$,*

$$D_\sigma = \{z \in \mathbb{C} : \ |\text{Im}\, z| < \sigma\}, \quad \sigma > 0,$$

*except the points $z = t + kT$, $k = 0, \pm 1, \pm 2, \ldots,$ where it has poles of order $m$. Then, for $m = 1, 2, 3, 4$ and all relevant $s$, there holds*

$$\left|\widehat{T}^{(s)}_{m,n}[f] - I[f]\right| \leq M_s(\rho) e^{-2n\pi\rho/T} \quad \forall \rho < \sigma,$$

*for some finite $M_s(\rho)$ that depends only on $f(z)$ and $s$.*

***Remarks*** 1. It is easy to see from this theorem that, for all practical purposes,

$$\widehat{T}^{(s)}_{m,n}[f] - I[f] = O\big(e^{-2n\pi\sigma/T}\big) \quad \text{as}\, n \to \infty.$$

2. Even though Theorem 1.2 concerns only the cases $m = 1, 2, 3, 4$, we conjecture that it holds true for every $m$. We invite the interested reader to try to prove this conjecture. The appendix to this work should be of help in extending Theorem 1.2 to $m = 5, 6, \ldots$.
3. We refer the reader to the numerical examples in [10] that clearly exhibit the exponential convergence of the $\widehat{T}^{(s)}_{m,n}[f]$ when $f(z)$ is $T$-periodic and analytic in a strip $D_\sigma$ of the complex plane.

## 2 Preliminaries to the proof of Theorem 1.2

We prove Theorem 1.2 in three stages:

1. We first show that, for each $m$, $I[f]$ can be expressed as a *regular* integral $\int_0^T \phi(x)\, dx$, where $\phi(x)$ is a $T$-periodic function on $\mathbb{R}$ that can be continued

analytically to the entire set $D_\sigma$ and is $T$-periodic and analytic in $D_\sigma$. We actually construct the function $\phi(x)$ explicitly.

2. We next show that the quadrature formula $\widehat{T}^{(0)}_{m,n}[f]$ for $I[f]$ is *identical to* the classical $n$-point trapezoidal rule $T_n[\phi] = h\sum_{j=0}^{n-1}\phi(jh)$, $h = T/n$, for the integral $\int_0^T \phi(x)\,dx$. This allows us to conclude that $\widehat{T}^{(0)}_{m,n}[f]$ converges to $I[f]$ exponentially in $n$; that is,

$$\left|\widehat{T}^{(0)}_{m,n}[f] - I[f]\right| \leq M_0(\rho)e^{-2n\pi\rho/T} \quad \forall\rho < \sigma.$$

We note that this result is based on the following theorem by Davis [3]:

**Theorem 2.1** *Let $\phi(z)$ be $T$-periodic and analytic in a strip $D_\sigma$ of the $z$-plane, where*

$$D_\sigma = \{z \in \mathbb{C} :\ |\mathrm{Im}\, z| < \sigma\}.$$

*Let*

$$I[\phi] = \int_0^T \phi(x)\,dx; \quad T_n[\phi] = h\sum_{j=0}^{n-1}\phi(jh), \quad h = \frac{T}{n}.$$

*Then, there holds*

$$\left|T_n[\phi] - I[\phi]\right| \leq T\,W(\rho)\frac{e^{-2n\pi\rho/T}}{1 - e^{-2n\pi\rho/T}} \quad \forall\rho < \sigma,$$

*where*

$$W(\rho) = \max_{x\in\mathbb{R}}\left|\phi(x + \mathrm{i}\rho)\right| + \max_{x\in\mathbb{R}}\left|\phi(x - \mathrm{i}\rho)\right|.$$

3. Following the treatment of $\widehat{T}^{(0)}_{m,n}[f]$, we invoke (1.7) to conclude that each $\widehat{T}^{(s)}_{m,n}[f]$, $s \geq 1$, also converges to $I[f]$ exponentially in $n$, and at the *same* rate as $\widehat{T}^{(0)}_{m,n}[f]$. The idea is that, by (1.7),

$$\widehat{T}^{(s)}_{m,n}[f] - I[f] = \sum_{k=0}^{s}\alpha_k^{(s)}\left(\widehat{T}^{(0)}_{m,2^k n}[f] - I[f]\right) \quad \text{because} \sum_{k=0}^{s}\alpha_k^{(s)} = 1,$$

from which we obtain

$$\begin{aligned}\left|\widehat{T}^{(s)}_{m,n}[f] - I[f]\right| &\leq \sum_{k=0}^{s}\left|\alpha_k^{(s)}\right|\left|\widehat{T}^{(0)}_{m,2^k n}[f] - I[f]\right| \\ &\leq \left(\sum_{k=0}^{s}\left|\alpha_k^{(s)}\right|\right)M_0(\rho)e^{-2n\pi\rho/T}\end{aligned}$$

that is stated in Theorem 1.2. In view of Theorem 2.1, we identify $M_0(\rho)$ as $TW(\rho)$ and $M_s(\rho)$ as $\left(\sum_{k=0}^{s}\left|\alpha_k^{(s)}\right|\right)M_0(\rho)$. As this completes the proof of Theorem 1.2, in the sequel, we need to carry out only the first and second stages mentioned above.

The first of the three stages above can be achieved in a simple way for $m = 1$, as we will see in Sect. 3. For $m \geq 2$, however, this stage demands some clever construction. The problem here is that we cannot make use of the representation $f(x) = g(x)/(x-t)^m$ in a direct way to construct $\phi(x)$, the reason being that neither $g(x)$ nor $(x-t)^m$ is $T$-periodic, and this does not allow us to proceed. Therefore, we aim to express $f(x)$ differently as $f(x) = u(x)v_m(x)$, such that,

1. as functions of the complex variable $z$, both $u(z)$ and $v_m(z)$ are $T$-periodic in $D_\sigma$,
2. $u(z)$ is analytic in $D_\sigma$,
3. $v_m(z)$ is nonzero and meromorphic in $D_\sigma$ with poles of order $m$ at $z = t + kT$, $k = 0, \pm 1, \pm 2, \ldots$.[3]

We choose

$$v_m(z) = \begin{cases} \dfrac{\exp\left(\mathrm{i}\frac{\pi(z-t)}{T}\right)}{\sin^m \frac{\pi(z-t)}{T}}, & m = 1, 3, 5, \ldots, \\[2ex] \dfrac{1}{\sin^m \frac{\pi(z-t)}{T}}, & m = 2, 4, 6, \ldots. \end{cases} \tag{2.1}$$

Note that $g(x)$ defines $u(x)$ uniquely and vice versa.[4]

To achieve the second stage above, we start by expressing $I[f] = \;=\!\!\!\!\!\!\int_a^b f(x)\,dx$ as the HFP integral of another function as follows: Since $f(x)$ is $T$-periodic, there holds

$$I[f] = \;=\!\!\!\!\!\!\!\int_t^{t+T} f(x)\,dx \quad \Rightarrow \quad I[f] = \;=\!\!\!\!\!\!\!\int_0^T f(t+y)\,dy.$$

Again, by $T$-periodicity,

$$I[f] = \;=\!\!\!\!\!\!\!\int_{t-T}^{t} f(x)\,dx \quad \Rightarrow \quad I[f] = \;=\!\!\!\!\!\!\!\int_0^T f(t-y)\,dy.$$

Therefore,

$$I[f] = \;=\!\!\!\!\!\!\!\int_0^T \frac{1}{2}\big[f(t+y) + f(t-y)\big]\,dy. \tag{2.2}$$

Following this, we replace $f(x)$ by its representation as $u(x)v_m(x)$ and proceed to the construction of $\phi(x)$. (We will show how this is done for each $m$ in Sects. 3–6.)

[3] If $v_m(z)$ vanishes at some point in $D_\sigma$ but $f(z)$ does not, then $u(z)$ must have a pole at that same point, which is not consistent with our demand that $u(z)$ be analytic in $D_\sigma$.

[4] Observe that (i) when $m$ is an even integer, $v_m(x)$ is real-valued, while (ii) when $m$ is an odd integer, $v_m(x)$ is complex-valued. Consequently, when $f(x)$ is a real-valued function, (i) $u(x)$ is real-valued if $m$ is an even integer, while (ii) $u(x)$ is complex-valued if $m$ is an odd integer.

Next, the trapezoidal sum

$$h\sum_{j=1}^{n-1}\frac{1}{2}\big[f(t+jh)+f(t-jh)\big],\quad h=\frac{T}{n}$$

for the HFP integral in (2.2) will feature in all our proofs. By $T$-periodicity of $f(x)$, we have

$$\sum_{j=1}^{n-1}\frac{1}{2}\big[f(t+jh)+f(t-jh)\big]=\sum_{j=1}^{n-1}\frac{1}{2}\big[f(t+jh)+f(t-jh+T)\big],$$

which, by the fact that $T=nh$, can be rewritten as

$$\sum_{j=1}^{n-1}\frac{1}{2}\big[f(t+jh)+f(t-jh)\big]=\sum_{j=1}^{n-1}\frac{1}{2}\big[f(t+jh)+f(t+(n-j)h)\big].$$

But $\{f(t+(n-j)h)\}_{j=1}^{n-1}=\{f(t+jh)\}_{j=1}^{n-1}$. Therefore,

$$\sum_{j=1}^{n-1}\frac{1}{2}\big[f(t+jh)+f(t-jh)\big]=\sum_{j=1}^{n-1}f(t+jh),\quad h=\frac{T}{n}. \tag{2.3}$$

We make repeated use of this in our proofs.

In addition, we shall make use of the HFP integrals[5]

$$\,=\!\!\!\!\!\!\int_0^T\frac{dy}{\sin^{2r}\frac{\pi y}{T}}=0,\quad r=1,2,\ldots, \tag{2.4}$$

and also of the summation rules

$$\sum_{j=1}^{n-1}\frac{1}{\sin^2\frac{j\pi}{n}}=\frac{n^2-1}{3},\quad \sum_{j=1}^{n-1}\frac{1}{\sin^4\frac{j\pi}{n}}=\frac{n^4+10n^2-11}{45}, \tag{2.5}$$

which can be found in Gradshteyn and Ryzhik [4, §15.823(3d)], for example.[6]

In conclusion, what remains now is

1. to construct a $T$-periodic function $\phi(x)$ that is integrable in the regular sense such that $\int_0^T\phi(x)\,dx=\,=\!\!\!\!\!\!\int_0^T f(x)\,dx$,
2. to show that, as a function of the complex variable $z$, $\phi(z)$ is $T$-periodic and analytic in $D_\sigma$ as well,

[5] See the appendix for a proof of this fact.

[6] See the appendix for the treatment of the general case of $\sum_{j=1}^{n-1}1/(\sin\frac{j\pi}{n})^{2r}$, which turns out to be a polynomial in $n^2$ of degree $r$, for $r=1,2,\ldots$, with rational coefficients.

3. to construct $T_n[\phi]$, the trapezoidal rule approximation for $\int_0^T \phi(x)\,dx$, and
4. to prove that $T_n[\phi] = \widehat{T}^{(0)}_{m,n}[f]$.

Clearly, in order to achieve all this, we also need to re-express $\widehat{T}^{(0)}_{m,n}[f]$ in (1.9a)–(1.12a) by expressing the "correction" terms involving $g^{(i)}(t)$ in terms of the appropriate $u^{(j)}(t)$. We actually have

$$g(t) = \left(\frac{T}{\pi}\right)^m u(t) \quad \forall\, m = 1, 2, \dots,$$
$$g^{(i)}(t) = \sum_{j=0}^{i} \alpha_{i,j}\, u^{(j)}(t), \quad i \geq 1, \quad \alpha_{i,j} \text{ constants independent of } t. \tag{2.6}$$

For $m = 1, 2, 3, 4$, these are given in (3.8), (4.3), (5.3), and (6.3).

## 3 Proof of Theorem 1.2 for $m = 1$

Invoking $f(x) = g(x)/(x - t)$ in (2.2), we obtain

$$I[f] = \oint_0^T \phi(y)\,dy, \quad \phi(y) = \frac{f(t+y) + f(t-y)}{2} = \frac{g(t+y) - g(t-y)}{2y}. \tag{3.1}$$

Now, $\phi(y)$ is $T$-periodic and regular for all $y \in (0, T)$ because $f(x) \in C^\infty(0, T)$. $\phi(y)$ is regular also at $y = 0$ (and at $y = T$ by $T$-periodicity) since

$$\phi(0) = \lim_{y\to 0} \phi(y) = g'(t). \tag{3.2}$$

Therefore, as a function of the complex variable $z$, $\phi(z)$ is analytic and $T$-periodic in $D_\sigma$. Consequently, $\oint_0^T \phi(y)\,dy$ is actually a regular integral.

Applying the trapezoidal rule to the regular integral $\int_0^T \phi(y)\,dy$ and invoking (3.2), we first obtain

$$T_n[\phi] = h\sum_{j=0}^{n-1} \phi(jh) = h\sum_{j=1}^{n-1} \phi(jh) + g'(t)h, \quad h = \frac{T}{n}. \tag{3.3}$$

Next, by (3.1),

$$\sum_{j=1}^{n-1} \phi(jh) = \sum_{j=1}^{n-1} \frac{g(t+jh) - g(t-jh)}{2jh} = \sum_{j=1}^{n-1} \frac{1}{2}[f(t+jh) + f(t-jh)],$$

which, by (2.3), becomes

$$\sum_{j=1}^{n-1}\phi(jh)=\sum_{j=1}^{n-1}f(t+jh). \tag{3.4}$$

Substituting this into (3.3), we finally have

$$T_n[\phi]=h\sum_{j=1}^{n-1}f(t+jh)+g'(t)h, \tag{3.5}$$

hence $T_n[\phi]=\widehat{T}^{(0)}_{1,n}[f]$ by (1.9a).

To complete the picture, we would like to express $g'(t)$ that appears in (1.9a) in terms of the appropriate $u^{(i)}(t)$, even though we were able to manage without this in our proof. Starting with

$$f(x)=\frac{\exp(\mathrm{i}\frac{\pi(x-t)}{T})}{\sin\frac{\pi(x-t)}{T}}u(x), \tag{3.6}$$

we relate $g(x)$ to $u(x)$ via

$$g(x)=(x-t)\frac{\exp(\mathrm{i}\frac{\pi(x-t)}{T})}{\sin\frac{\pi(x-t)}{T}}u(x). \tag{3.7}$$

Letting $\xi=\pi(x-t)/T$ for short and noting that

$$\xi\frac{e^{\mathrm{i}\xi}}{\sin\xi}=1+\mathrm{i}\xi+O(\xi^2)\quad \text{as}\,\xi\to 0,$$

and expanding the right-hand side of (3.7) about $x=t$, we have

$$g(x)=\frac{T}{\pi}\big[1+\mathrm{i}\xi+O(\xi^2)\big]\big[u(t)+u'(t)(x-t)+O(\xi^2)\big]\quad \text{as}\,x\to t.$$

Thus we identify

$$g(t)=\frac{T}{\pi}u(t),\quad g'(t)=\mathrm{i}u(t)+\frac{T}{\pi}u'(t). \tag{3.8}$$

With this, (1.9a) becomes

$$\boxed{\widehat{T}^{(0)}_{1,n}[f]=h\sum_{j=1}^{n-1}f(t+jh)+\mathrm{i}u(t)h+\frac{T}{\pi}u'(t)h.} \tag{3.9}$$

## 4 Proof of Theorem 1.2 for $m = 2$

Let us express $f(x)$ as

$$f(x) = \frac{u(x)}{\sin^2 \frac{\pi(x-t)}{T}}. \tag{4.1}$$

Thus, $g(x)$ is related to $u(x)$ via

$$g(x) = (x-t)^2 f(x) = \frac{(x-t)^2}{\sin^2 \frac{\pi(x-t)}{T}} u(x). \tag{4.2}$$

Clearly, $g(x)$ is guaranteed to be infinitely differentiable on $(a, b)$ when $a < t < b$.

We start by expressing $g(t)$ and $g''(t)$ in (1.10a) in terms of $u(x)$ and derivatives of $u(x)$ at $x = t$. Letting $\xi = \pi(x-t)/T$ for short and noting that

$$\frac{\xi^2}{\sin^2 \xi} = 1 + \frac{\xi^2}{3} + O(\xi^4) \quad \text{as}\, \xi \to 0,$$

we expand the right-hand side of (4.2) about $x = t$. We have

$$g(x) = \left(\frac{T}{\pi}\right)^2 \left[1 + \frac{\xi^2}{3} + O(\xi^4)\right] \left[\sum_{k=0}^{2} \frac{u^{(k)}(t)}{k!}(x-t)^k + O(\xi^3)\right] \quad \text{as}\, x \to t.$$

We then identify

$$g(t) = \left(\frac{T}{\pi}\right)^2 u(t), \quad g'(t) = \left(\frac{T}{\pi}\right)^2 u'(t), \quad g''(t) = \left(\frac{T}{\pi}\right)^2 u''(t) + \frac{2}{3}u(t). \tag{4.3}$$

Substituting these into (1.10a), and invoking also $T = nh$, we obtain

$$\boxed{\widehat{T}^{(0)}_{2,n}[f] = h\sum_{j=1}^{n-1} f(t+jh) - \left(\frac{n^2-1}{3}\right)u(t)h + \frac{1}{2}\left(\frac{T}{\pi}\right)^2 u''(t)h.} \tag{4.4}$$

We next show that $I[f]$ can be expressed as a regular integral $\int_0^T \phi(y)\,dy$, $\phi(z)$ being a $T$-periodic function that is analytic in the strip $D_\sigma$. Substituting (4.1) into (2.2), and simplifying, we have

$$I[f] = \ddagger\!\!\!\!\int_0^T \left[\frac{u(t+y) + u(t-y)}{2\sin^2 \frac{\pi y}{T}}\right] dy,$$

which continues to be defined only in the sense of HFP. Upon subtracting $2u(t)$ from the numerator of the integrand (and adding it back), we obtain

$$I[f] = \neq\!\!\!\!\!\!\int_0^T \left[\frac{u(t+y)+u(t-y)-2u(t)}{2\sin^2\frac{\pi y}{T}}\right] dy + u(t) \neq\!\!\!\!\!\!\int_0^T \frac{1}{\sin^2\frac{\pi y}{T}}\, dy.$$

Since $\neq\!\!\!\!\!\!\int_0^T 1/(\sin^2\frac{\pi y}{T})\, dy = 0$, we have

$$I[f] = \neq\!\!\!\!\!\!\int_0^T \phi(y)\, dy, \quad \phi(y) = \frac{u(t+y)+u(t-y)-2u(t)}{2\sin^2\frac{\pi y}{T}}. \tag{4.5}$$

Now, $\phi(y)$ is $T$-periodic and regular for all $y \in (0, T)$ because $f(x) \in C^\infty(0, T)$. $\phi(y)$ is regular also at $y = 0$ (and at $y = T$ by $T$-periodicity) since

$$\phi(0) = \lim_{y\to 0} \phi(y) = \frac{1}{2}\left(\frac{T}{\pi}\right)^2 u''(t), \tag{4.6}$$

which can be obtained by two applications of L'Hôpital's rule or by simply expanding $u(t+y)+u(t-y)-2u(t)$ in a Taylor series about $y = 0$. Therefore, as a function of the complex variable $z$, $\phi(z)$ is analytic and $T$-periodic in $D_\sigma$. Consequently, $\neq\!\!\!\!\!\!\int_0^T \phi(y)\, dy$ is actually a regular integral.

Applying the trapezoidal rule to the regular integral $\int_0^T \phi(y)\, dy$ and invoking (4.6), we first obtain

$$T_n[\phi] = h\sum_{j=0}^{n-1} \phi(jh) = h\sum_{j=1}^{n-1} \phi(jh) + \frac{1}{2}\left(\frac{T}{\pi}\right)^2 u''(t)h, \quad h = \frac{T}{n}. \tag{4.7}$$

Next, by (4.5),

$$\sum_{j=1}^{n-1} \phi(jh) = \sum_{j=1}^{n-1} \frac{u(t+jh)+u(t-jh)}{2\sin^2\frac{j\pi}{n}} - u(t)\sum_{j=1}^{n-1} \frac{1}{\sin^2\frac{j\pi}{n}}.$$

Now, by (4.1) and (2.3),

$$\sum_{j=1}^{n-1} \frac{u(t+jh)+u(t-jh)}{2\sin^2\frac{j\pi}{n}} = \sum_{j=1}^{n-1} \frac{1}{2}[f(t+jh)+f(t-jh)] = \sum_{j=1}^{n-1} f(t+jh).$$

Combining these in (4.7), and invoking also (2.5), we obtain

$$T_n[\phi] = h\sum_{j=1}^{n-1} f(t+jh) - \left(\frac{n^2-1}{3}\right)u(t)h + \frac{1}{2}\left(\frac{T}{\pi}\right)^2 u''(t)h. \tag{4.8}$$

Comparing (4.8) with (4.4), we realize that $T_n[\phi] = \widehat{T}^{(0)}_{2,n}[f]$.

## 5 Proof of Theorem 1.2 for $m = 3$

Let us express $f(x)$ as

$$f(x) = \frac{\exp(\mathrm{i}\frac{\pi(x-t)}{T})}{\sin^3 \frac{\pi(x-t)}{T}} u(x). \tag{5.1}$$

Thus, $g(x)$ is related to $u(x)$ via

$$g(x) = (x-t)^3 f(x) = \frac{(x-t)^3 \exp(\mathrm{i}\frac{\pi(x-t)}{T})}{\sin^3 \frac{\pi(x-t)}{T}} u(x). \tag{5.2}$$

Clearly, $g(x)$ is guaranteed to be infinitely differentiable on $(a, b)$ when $a < t < b$.

We start by expressing $g'(t)$ and $g'''(t)$ in (1.11a) in terms of $u(x)$ and derivatives of $u(x)$ at $x = t$. Letting $\xi = \pi(x-t)/T$ for short and noting that

$$\frac{\xi^3 e^{\mathrm{i}\xi}}{\sin^3 \xi} = 1 + \mathrm{i}\xi + \mathrm{i}\frac{\xi^3}{3} + O(\xi^4) \quad \text{as } \xi \to 0,$$

we expand the right-hand side of (5.2) about $x = t$. We have

$$g(x) = \left(\frac{T}{\pi}\right)^3 \left[1 + \mathrm{i}\xi + \mathrm{i}\frac{\xi^3}{3} + O(\xi^4)\right]\left[\sum_{k=0}^{3} \frac{u^{(k)}(t)}{k!}(x-t)^k + O(\xi^4)\right] \quad \text{as } x \to t.$$

We then identify

$$\begin{aligned}
g(t) &= \left(\frac{T}{\pi}\right)^3 u(t),\\
g'(t) &= \left(\frac{T}{\pi}\right)^3 \left[u'(t) + \mathrm{i}\left(\frac{\pi}{T}\right)u(t)\right],\\
g''(t) &= \left(\frac{T}{\pi}\right)^3 \left[u''(t) + \mathrm{i}\left(\frac{\pi}{T}\right)2u'(t)\right],\\
g'''(t) &= \left(\frac{T}{\pi}\right)^3 \left[u'''(t) + \mathrm{i}\left(\frac{\pi}{T}\right)3u''(t) + \mathrm{i}\left(\frac{\pi}{T}\right)^3 2u(t)\right].
\end{aligned} \tag{5.3}$$

Following this, we also rewrite (5.1) in the form

$$f(x) = f_1(x) + \mathrm{i} f_2(x); \quad f_1(x) = \frac{\cos \frac{\pi(x-t)}{T}}{\sin^3 \frac{\pi(x-t)}{T}} u(x), \quad f_2(x) = \frac{u(x)}{\sin^2 \frac{\pi(x-t)}{T}}. \tag{5.4}$$

Therefore, we also have

$$I[f]=I[f_1]+\mathrm{i}I[f_2]. \tag{5.5}$$

Substituting (5.3) and (5.4) into (1.11a), and invoking also $T=nh$, we obtain

$$\widehat{T}^{(0)}_{3,n}[f]=Q_1+\mathrm{i}Q_2, \tag{5.6}$$

where

$$Q_1=h\sum_{j=1}^{n-1}f_1(t+jh)-\frac{n^2}{3}\left(\frac{T}{\pi}\right)u'(t)h+\frac{1}{6}\left(\frac{T}{\pi}\right)^3u'''(t)h, \tag{5.7}$$

$$Q_2=h\sum_{j=1}^{n-1}f_2(t+jh)-\left(\frac{n^2-1}{3}\right)u(t)h+\frac{1}{2}\left(\frac{T}{\pi}\right)^2u''(t)h. \tag{5.8}$$

Therefore,

$$\boxed{\begin{aligned}\widehat{T}^{(0)}_{3,n}[f]=h\sum_{j=1}^{n-1}f(t+jh)-\mathrm{i}\left(\frac{n^2-1}{3}\right)u(t)h-\frac{n^2}{3}\left(\frac{T}{\pi}\right)u'(t)h\\+\mathrm{i}\frac{1}{2}\left(\frac{T}{\pi}\right)^2u''(t)h+\frac{1}{6}\left(\frac{T}{\pi}\right)^3u'''(t)h.\end{aligned}} \tag{5.9}$$

We now turn to the study of $I[f_1]=\displaystyle\neq\!\!\!\!\!\!\int_a^b f_1(x)\,dx$ and $I[f_2]=\neq\!\!\!\!\!\!\int_a^b f_2(x)\,dx$. Applying (2.2) to $f_1(x)$, we first have

$$I[f_1]=\neq\!\!\!\!\!\!\!\int_0^T\frac{\cos\frac{\pi y}{T}}{2\sin^3\frac{\pi y}{T}}[u(t+y)-u(t-y)]\,dy,$$

which continues to be defined only in the sense of HFP. Upon subtracting $2(\frac{T}{\pi})u'(t)\tan\frac{\pi y}{T}$ from the square brackets (and adding it back), we obtain

$$\begin{aligned}I[f_1]=&\neq\!\!\!\!\!\!\!\int_0^T\frac{\cos\frac{\pi y}{T}}{2\sin^3\frac{\pi y}{T}}\left[u(t+y)-u(t-y)-2\left(\frac{T}{\pi}\right)u'(t)\tan\left(\frac{\pi y}{T}\right)\right]dy\\&+\frac{T}{\pi}u'(t)\neq\!\!\!\!\!\!\!\int_0^T\frac{1}{\sin^2\frac{\pi y}{T}}\,dy.\end{aligned}$$

Since $\neq\!\!\!\!\!\!\int_0^T 1/(\sin^2\frac{\pi y}{T})\,dy=0$, we have

$$I[f_1]=\neq\!\!\!\!\!\!\!\int_0^T\phi_1(y)\,dy,\quad \phi_1(y)=\frac{\cos\frac{\pi y}{T}}{2\sin^3\frac{\pi y}{T}}\left[u(t+y)-u(t-y)-2\left(\frac{T}{\pi}\right)u'(t)\tan\frac{\pi y}{T}\right]. \tag{5.10}$$

Now, $\phi_1(y)$ is $T$-periodic and regular for all $y \in (0, T)$ because $f(x) \in C^\infty(0, T)$. $\phi_1(y)$ is regular also at $y = 0$ (and at $y = T$ by $T$-periodicity) since

$$\phi_1(0) = \lim_{y\to 0} \phi_1(y) = \frac{1}{6}\left(\frac{T}{\pi}\right)^3 u'''(t) - \frac{1}{3}\left(\frac{T}{\pi}\right)u'(t), \tag{5.11}$$

which can be obtained from (5.10) by noting that

$$\frac{\cos\xi}{\sin^3\xi} = \xi^{-3}\big[1 + O(\xi^4)\big] \quad \text{and} \quad \tan\xi = \xi\left[1 + \frac{\xi^2}{3} + O(\xi^4)\right] \quad \text{as } \xi \to 0$$

and

$$u(t + y) - u(t - y) = 2u'(t)y + \frac{1}{3}u'''(t)y^3 + O(y^5) \quad \text{as } y \to 0.$$

Therefore, as a function of the complex variable $z$, $\phi_1(z)$ is analytic and $T$-periodic in $D_\sigma$. Consequently, $\displaystyle{\int\!\!\!\!\!\!=}_0^T \phi_1(y)\,dy$ is actually a regular integral.

Applying the trapezoidal rule to the regular integral $\int_0^T \phi_1(y)\,dy$ and invoking (5.11), we first obtain

$$T_n[\phi_1] = h\sum_{j=0}^{n-1}\phi_1(jh) = h\sum_{j=1}^{n-1}\phi_1(jh) - \frac{1}{3}\left(\frac{T}{\pi}\right)u'(t)h + \frac{1}{6}\left(\frac{T}{\pi}\right)^3 u'''(t)h, \quad h = \frac{T}{n}. \tag{5.12}$$

Next, by (4.5),

$$\sum_{j=1}^{n-1}\phi_1(jh) = \sum_{j=1}^{n-1}\frac{\cos\frac{j\pi}{n}}{2\sin^3\frac{j\pi}{n}}\big[u(t + jh) - u(t - jh)\big] - \left(\frac{T}{\pi}\right)u'(t)\sum_{j=1}^{n-1}\frac{1}{\sin^2\frac{j\pi}{n}}.$$

Now, by (2.3),

$$\begin{aligned}\sum_{j=1}^{n-1}\frac{\cos\frac{j\pi}{n}}{2\sin^3\frac{j\pi}{n}}\big[u(t + jh) - u(t - jh)\big] &= \sum_{j=1}^{n-1}\frac{1}{2}\big[f_1(t + jh) + f_1(t - jh)\big]\\ &= \sum_{j=1}^{n-1} f_1(t + jh).\end{aligned}$$

By this and by (2.5), we thus have that

$$\sum_{j=1}^{n-1}\phi_1(jh) = \sum_{j=1}^{n-1} f_1(t + jh) - \left(\frac{n^2 - 1}{3}\right)\left(\frac{T}{\pi}\right)u'(t).$$

Therefore,

$$T_n[\phi_1] = h\sum_{j=1}^{n-1} f_1(t+jh) - \frac{n^2}{3}\left(\frac{T}{\pi}\right)u'(t)h + \frac{1}{6}\left(\frac{T}{\pi}\right)^3 u'''(t)h.$$

Comparing this with (5.7), we realize that $T_n[\phi_1] = Q_1$.

As for $I[f_2]$, we already know from Sect. 4 that

$$I[f_2] = \int_0^T \phi_2(y)\,dy, \quad \phi_2(y) = \frac{u(t+y)+u(t-y)-2u(t)}{2\sin^2\frac{\pi y}{T}},$$

and $T_n[\phi_2] = \widehat{T}^{(0)}_{2,n}[f_2]$. In addition, by (5.8) and (4.8), $Q_2 = \widehat{T}^{(0)}_{2,n}[f_2]$, hence $Q_2 = T_n[\phi_2]$ as well. Therefore, letting $\phi(y) = \phi_1(y) + \mathrm{i}\phi_2(y)$, and recalling (5.9), we have

$$\int_0^T \phi(y)\,dy = I[f] \quad\text{and}\quad T_n[\phi] = T_n[\phi_1] + \mathrm{i}T_n[\phi_2] = Q_1 + \mathrm{i}Q_2 = \widehat{T}^{(0)}_{3,n}[f].$$

## 6 Proof of Theorem 1.2 for $m = 4$

Let us express $f(x)$ as

$$f(x) = \frac{u(x)}{\sin^4\frac{\pi(x-t)}{T}}. \tag{6.1}$$

Thus, $g(x)$ is related to $u(x)$ via

$$g(x) = (x-t)^4 f(x) = \frac{(x-t)^4}{\sin^4\frac{\pi(x-t)}{T}}u(x). \tag{6.2}$$

Clearly, $g(x)$ is guaranteed to be infinitely differentiable on $(a, b)$ when $a < t < b$.

We start by expressing $g(t)$, $g''(t)$, and $g^{(4)}(t)$ in (1.12a) in terms of $u(x)$ and derivatives of $u(x)$ at $x = t$. Letting $\xi = \pi(x-t)/T$ for short and noting that

$$\frac{\xi^4}{\sin^4\xi} = 1 + \frac{2}{3}\xi^2 + \frac{11}{45}\xi^4 + O(\xi^6) \quad \text{as}\ \xi \to 0,$$

we expand the right-hand side of (6.2) about $x = t$. We have

$$g(x) = \left(\frac{T}{\pi}\right)^4\left[1 + \frac{2}{3}\xi^2 + \frac{11}{45}\xi^4 + O(\xi^6)\right]\left[\sum_{k=0}^{4}\frac{u^{(k)}(t)}{k!}(x-t)^k + O(\xi^5)\right] \quad \text{as}\ x \to t.$$

We then identify

$$\begin{aligned}
g(t) &= \left(\frac{T}{\pi}\right)^4 u(t),\\
g'(t) &= \left(\frac{T}{\pi}\right)^4 u'(t),\\
g''(t) &= \left(\frac{T}{\pi}\right)^4 \left[u''(t) + \frac{4}{3}\left(\frac{\pi}{T}\right)^2 u(t)\right],\\
g'''(t) &= \left(\frac{T}{\pi}\right)^4 \left[u'''(t) + 4\left(\frac{\pi}{T}\right)^2 u'(t)\right],\\
g^{(4)}(t) &= \left(\frac{T}{\pi}\right)^4 \left[u^{(4)}(t) + 8\left(\frac{\pi}{T}\right)^2 u''(t) + \frac{88}{15}\left(\frac{\pi}{T}\right)^4 u(t)\right].
\end{aligned} \tag{6.3}$$

Substituting these into (1.12a), we first obtain

$$\begin{aligned}
\widehat{T}^{(0)}_{4,n}[f] = h\sum_{j=1}^{n-1} f(t+jh) &+ \left(-\frac{T^4}{45}h^{-3} - \frac{2T^2}{9}h^{-1} + \frac{11}{45}h\right)u(t)\\
&+ \left(-\frac{T^4}{6\pi^2}h^{-1} + \frac{T^2}{3\pi^2}h\right)u''(t) + \left(\frac{T^4}{24\pi^4}h\right)u^{(4)}(t),
\end{aligned}$$

which, upon invoking $T = nh$ where necessary, becomes

$$\boxed{\begin{aligned}
\widehat{T}^{(0)}_{4,n}[f] = h\sum_{j=1}^{n-1} f(t+jh) &- \left(\frac{n^4 + 10n^2 - 11}{45}\right)u(t)h\\
&- \left(\frac{n^2-2}{6}\right)\left(\frac{T}{\pi}\right)^2 u''(t)h + \frac{1}{24}\left(\frac{T}{\pi}\right)^4 u^{(4)}(t)h.
\end{aligned}} \tag{6.4}$$

We now turn to the study of $I[f]$. Applying (2.2), we first have

$$I[f] = \,=\!\!\!\!\!\!\int_0^T \frac{u(t+y) + u(t-y)}{2\sin^4 \frac{\pi y}{T}}\, dy,$$

which continues to be defined only in the sense of HFP. Upon subtracting $2[u(t) + \frac{1}{2}(\frac{T}{\pi})^2 u''(t)\sin^2\frac{\pi y}{T}]$ from the numerator of the integrand (and adding it back), we obtain

$$I[f] = \fint_0^T \frac{u(t+y) + u(t-y) - 2u(t) - (\frac{T}{\pi})^2 u''(t) \sin^2 \frac{\pi y}{T}}{2 \sin^4 \frac{\pi y}{T}} \, dy$$
$$+ u(t) \fint_0^T \frac{dy}{\sin^4 \frac{\pi y}{T}} + \frac{1}{2} \left(\frac{T}{\pi}\right)^2 u''(t) \fint_0^T \frac{dy}{\sin^2 \frac{\pi y}{T}}.$$

Since $\fint_0^T dy/(\sin^2 \frac{\pi y}{T}) = 0$ and $\fint_0^T dy/(\sin^4 \frac{\pi y}{T}) = 0$, we have

$$I[f] = \fint_0^T \phi(y) \, dy, \quad \phi(y) = \frac{u(t+y) + u(t-y) - 2u(t) - (\frac{T}{\pi})^2 u''(t) \sin^2 \frac{\pi y}{T}}{2 \sin^4 \frac{\pi y}{T}}. \tag{6.5}$$

Now, $\phi(y)$ is $T$-periodic and regular for all $y \in (0, T)$ because $f(x) \in C^\infty(0, T)$. $\phi(y)$ is regular also at $y = 0$ (and at $y = T$ by $T$-periodicity) since

$$\phi(0) = \lim_{y \to 0} \phi(y) = \frac{1}{24} \left(\frac{T}{\pi}\right)^4 u^{(4)}(t) + \frac{1}{6} \left(\frac{T}{\pi}\right)^2 u''(t), \tag{6.6}$$

which can be obtained by simply expanding the numerator and denominator of $\phi(y)$ in a Taylor series about $y = 0$. Therefore, as a function of the complex variable $z$, $\phi(z)$ is analytic and $T$-periodic in $D_\sigma$. Consequently, $\fint_0^T \phi(y) \, dy$ is actually a regular integral.

Applying the trapezoidal rule to the regular integral $\int_0^T \phi(y) \, dy$ and invoking (6.6), we first obtain

$$T_n[\phi] = h \sum_{j=0}^{n-1} \phi(jh) = h \sum_{j=1}^{n-1} \phi(jh) + \frac{1}{6} \left(\frac{T}{\pi}\right)^2 u''(t) h + \frac{1}{24} \left(\frac{T}{\pi}\right)^4 u^{(4)}(t) h, \quad h = \frac{T}{n}. \tag{6.7}$$

Next, by (6.5),

$$\sum_{j=1}^{n-1} \phi(jh) = \sum_{j=1}^{n-1} \frac{u(t+jh) + u(t-jh)}{2 \sin^4 \frac{j\pi}{n}} - u(t) \sum_{j=1}^{n-1} \frac{1}{\sin^4 \frac{j\pi}{n}} - \frac{1}{2} \left(\frac{T}{\pi}\right)^2 u''(t) \sum_{j=1}^{n-1} \frac{1}{\sin^2 \frac{j\pi}{n}}.$$

Now, by (6.1) and (2.3),

$$\sum_{j=1}^{n-1} \frac{u(t+jh) + u(t-jh)}{2 \sin^4 \frac{j\pi}{n}} = \sum_{j=1}^{n-1} \frac{1}{2} [f(t+jh) + f(t-jh)] = \sum_{j=1}^{n-1} f(t+jh).$$

Substituting this into (6.7), and invoking also (2.5), we obtain

$$T_n[\phi] = h\sum_{j=1}^{n-1} f(t+jh) - \left(\frac{n^4+10n-11}{45}\right)u(t)h \\ - \left(\frac{n^2-2}{6}\right)\left(\frac{T}{\pi}\right)^2 u''(t)h + \frac{1}{24}\left(\frac{T}{\pi}\right)^4 u^{(4)}(t)h. \quad (6.8)$$

Comparing (6.8) with (6.4), we realize that $T_n[\phi] = \widehat{T}^{(0)}_{4,n}[f]$.

## Appendix

In this appendix, we describe some tools we believe should be helpful in extending the treatment we have presented for $m = 1, 2, 3, 4$ to arbitrary $m$.

### 1. Determination of $\phi(y)$

Our first tool concerns the $T$-periodic Taylor-like sum to be subtracted from the expressions $[u(t+y) \pm u(t-y)]$ in order to obtain a $T$-periodic and analytic $\phi(y)$.

- When dealing with

$$\frac{u(t+y)+u(t-y)}{2\sin^{2r}\frac{\pi y}{T}},$$

  subtract $(A_0 + \sum_{k=1}^{r-1} A_k \sin^{2k}\frac{\pi y}{T})$, which is $T$-periodic, from $[u(t+y)+u(t-y)]$, with $A_0, A_1, \ldots, A_{r-1}$ such that

$$\phi(y) = \frac{1}{2\sin^{2r}\frac{\pi y}{T}}\left[u(t+y)+u(t-y) - \left(A_0 + \sum_{k=1}^{r-1} A_k \sin^{2k}\frac{\pi y}{T}\right)\right]$$

  is well-defined at $y = 0$ (hence at $y = T$ as well). This means that the expression inside the square brackets must be $O(y^{2r})$ as $y \to 0$. The $A_k$ can be determined one by one in the order $A_0, A_1, \ldots$. For example, $A_0 = 2u(t)$, $A_1 = (\frac{T}{\pi})^2 u''(t)$, and so on.
- When dealing with

$$\frac{\cos\frac{\pi y}{T}}{2\sin^{2r+1}\frac{\pi y}{T}}[u(t+y)-u(t-y)]\,dy,$$

subtract $[\tan\frac{\pi y}{T}(B_0+\sum_{k=1}^{r-1} B_k \sin^{2k}\frac{\pi y}{T})]$, which is $T$-periodic, from $[u(t+y)-u(t-y)]$, with $B_0, B_1, \ldots, B_{r-1}$ such that

$$\phi(y)=\frac{\cos\frac{\pi y}{T}}{2\sin^{2r+1}\frac{\pi y}{T}}\left[u(t+y)-u(t-y)-\tan\frac{\pi y}{T}\left(B_0+\sum_{k=1}^{r-1} B_k \sin^{2k}\frac{\pi y}{T}\right)\right]$$

is well-defined at $y=0$ (hence at $y=T$ as well). This means that the expression inside the square brackets must be $O(y^{2r+1})$ as $y\to 0$. The $B_k$ can be determined one by one in the order $B_0, B_1, \ldots$. For example, $B_0=2(\frac{T}{\pi})u'(t)$, $B_1=\frac{1}{3}(\frac{T}{\pi})^3u'''(t)-\frac{2}{3}(\frac{T}{\pi})u'(t)$, and so on.

In both cases, invoke also (2.4). The end result is $I[f]=\int_0^T \phi(x)\,dx$.

## 2. Determination of $\sum_{j=1}^{n-1} 1/(\sin\frac{j\pi}{n})^{2r}, r=1,2,\ldots$

In the treatment of $\widehat{T}^{(0)}_{m,n}[f]$ with $m=2,3,4$, we made repeated use of (2.5). We can actually deal with arbitrary $m$ in exactly the same way we dealt with the cases $m=2,3,4$ provided we know $\sum_{j=1}^{n-1} 1/(\sin\frac{j\pi}{n})^{2r}$ analytically as a function of $n$. We turn precisely to this problem here. This is our second tool.

Let us consider the integral $\int_0^1 w_p(x)\,dx$, where $w_p(x)=1/(\sin\pi x)^p$, $p$ being complex and arbitrary. We begin by noting that, as a regular integral,

$$\int_0^1 \frac{1}{(\sin\pi x)^p}\,dx=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}p)}{\Gamma(1-\frac{1}{2}p)},\quad \operatorname{Re} p<1. \tag{A.1}$$

We next note that the right-hand side of (A.1) is analytic in $p$, except when $p=1,3,5,\ldots$, where it has simple poles. Thus, in case of divergence, that is, when $\operatorname{Re} p\geq 1$, the right-hand side of (A.1) is the HFP of $\int_0^1 w_p(x)\,dx$, provided $p\neq 1,3,5,\ldots$. That is,

$$\mathop{\not\!\!\int}_0^1 \frac{1}{(\sin\pi x)^p}\,dx=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}p)}{\Gamma(1-\frac{1}{2}p)},\quad \forall\, p\in\mathbb{C}\backslash\{1,3,5,\ldots\}. \tag{A.2}$$

Therefore, letting $p=2r$, $r=1,2,\ldots$, in (A.2) and recalling the fact that $1/\Gamma(-k)=0$, $k=0,1,2,\ldots$, we obtain

$$\mathop{\not\!\!\int}_0^1 \frac{1}{(\sin\pi x)^{2r}}\,dx=0,\quad r=1,2,3,\ldots. \tag{A.3}$$

In addition, $w_{2r}(x)$ is infinitely differentiable on $(0,1)$, and has the (convergent) asymptotic expansions

$$w_{2r}(x)=\sum_{s=0}^{\infty}\beta_s(r)x^{2s-2r},\quad |x|<1,\quad (\text{as } x\to 0),$$

and

$$w_{2r}(x) = \sum_{s=0}^{\infty} \beta_s(r)(1-x)^{2s-2r}, \quad |1-x| < 1, \quad (\text{as } x \to 1),$$

where $\beta_s(r)$ are defined in terms of the *generalized Bernoulli polynomials* $B_s^{(\sigma)}(u)$ as in

$$\beta_s(r) = (-1)^s 2^{2s} \pi^{2s-2r} \frac{B_{2s}^{(2r)}(r)}{(2s)!}, \quad s = 0, 1, \ldots.$$

Invoking Corollary 2.2 in [7] or Theorem 2.3 in [8], and recalling (A.3), we obtain

$$\sum_{j=1}^{n-1} \frac{1}{(\sin \frac{j\pi}{n})^{2r}} = \sum_{s=0}^{r} (-1)^s 2^{2s+1} \pi^{2s-2r} \frac{B_{2s}^{(2r)}(r)}{(2s)!} \zeta(2r-2s) n^{2r-2s}, \qquad \text{(A.4)}$$

first noted by Brauchart, Hardin, and Saff [2]. (See also the example in [8, Section 4].) Invoking also

$$\zeta(2k) = (-1)^{k-1} 2^{2k-1} \pi^{2k} \frac{B_{2k}}{(2k)!}, \quad k = 0, 1, 2, \ldots,$$

where $B_i$ are Bernoulli numbers, (A.4) simplifies to read

$$\begin{aligned}\sum_{j=1}^{n-1} \frac{1}{(\sin \frac{j\pi}{n})^{2r}} &= (-1)^{r-1} 4^r \sum_{s=0}^{r} \frac{B_{2r-2s}}{(2r-2s)!} \frac{B_{2s}^{(2r)}(r)}{(2s)!} n^{2r-2s} \\ &= (-1)^{r-1} \frac{4^r}{(2r)!} \sum_{s=0}^{r} \binom{2r}{2s} B_{2r-2s} B_{2s}^{(2r)}(r) n^{2r-2s}. \qquad \text{(A.5)}\end{aligned}$$

Note that the end result is a polynomial in $n^2$ of degree $r$. In addition, the coefficients of this polynomial are rational numbers since both $B_{2s}$ and $B_{2s}^{(2r)}(r)$ are rational numbers.

### 3. A brief introduction to generalized Bernoulli numbers and polynomials

The *generalized Bernoulli numbers* $B_s^{(\sigma)}$ and *generalized Bernoulli polynomials* $B_s^{(\sigma)}(u)$ are defined via (see Luke [5, p. 18–23] or Andrews, Askey, and Roy [1, p. 615], for example)

$$\left(\frac{t}{e^t-1}\right)^{\sigma} = \sum_{s=0}^{\infty} B_s^{(\sigma)} \frac{t^s}{s!} \quad \text{and} \quad \left(\frac{t}{e^t-1}\right)^{\sigma} e^{ut} = \sum_{s=0}^{\infty} B_s^{(\sigma)}(u) \frac{t^s}{s!}, \quad |t| < 2\pi.$$

$B_s^{(\sigma)}(u)$ is of degree $s$ in $u$ and is given as $B_s^{(\sigma)}(u)=\sum_{k=0}^{s}\binom{s}{k}B_{s-k}^{(\sigma)}u^k$ and $B_s^{(\sigma)}(0)=B_s^{(\sigma)}$. Note that, for all $\sigma$, $B_0^{(\sigma)}=1$ and $B_0^{(\sigma)}(u)\equiv 1$.

$B_s^{(\sigma)}$ is a polynomial in $\sigma$ of degree $s$, and is also known as a *Nörlund polynomial*. Note also that $B_s^{(\sigma)}$ are rational numbers when $\sigma$ is a rational number.

In addition, $B_s^{(\sigma)}(\sigma-u)=(-1)^sB_s^{(\sigma)}(u)$; therefore, for $s=0,1,2,\dots$, $B_{2s+1}^{(\sigma)}(\sigma/2)=0$ while $B_{2s}^{(\sigma)}(\sigma/2)$ is a polynomial in $\sigma$ of degree $s$. In addition, when $\sigma$ is a rational number $B_{2s}^{(\sigma)}(\sigma/2)$ is a rational number too.

For completeness, we give a few of the $B_{2s}^{(2r)}(r)$ needed in (A.5). For a longer list, see [5, p. 34]:

$$B_0^{(2r)}(r)=1,\quad B_2^{(2r)}(r)=-\frac{r}{6},\quad B_4^{(2r)}(r)=\frac{r(5r+1)}{60},$$
$$B_6^{(2r)}(r)=-\frac{r}{504}(35r^2+21r+4),\quad B_8^{(2r)}(r)=\frac{r}{2160}(175r^3+210r^2+101r+18).$$

By letting $r=1,2$ in (A.5), we obtain the summation rules in (2.5). To end, we use (A.5) to compute $\sum_{j=1}^{n-1}1/(\sin\frac{j\pi}{n})^{2r}$ for $r=3,4$:

$$\sum_{j=1}^{n-1}\frac{1}{(\sin\frac{j\pi}{n})^6}=\frac{1}{945}(2n^6+21n^4+168n^2-191),$$
$$\sum_{j=1}^{n-1}\frac{1}{(\sin\frac{j\pi}{n})^8}=\frac{1}{14175}(3n^8+40n^6+294n^4+2160n^2-2497).$$

## References


1. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)
2. Brauchart, J.S., Hardin, D.P., Saff, E.B.: The Riesz energy of the $N$th roots of unity: an asymptotic expansion for large $N$. Bull. Lond. Math. Soc. **41**, 621–633 (2009)
3. Davis, P.J.: On the numerical integration of periodic analytic functions. In: Langer, R.E., (eds.) Proceedings of a Symposium on Numerical Approximation, pp. 45–59. University of Wisconsin Press, Madison (1959)
4. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Academic Press, New York (2007)
5. Luke, Y.L.: The Special Functions and Their Approximations, vol. I. Academic Press, New York (1969)
6. Sidi, A.: Practical Extrapolation Methods: Theory and Applications. Number 10 in Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge (2003)
7. Sidi, A.: Euler–Maclaurin expansions for integrals with endpoint singularities: a new perspective. Numer. Math. **98**, 371–387 (2004)
8. Sidi, A.: Euler–Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities. Math. Comput. **81**, 2159–2173 (2012)
9. Sidi, A.: Compact numerical quadrature formulas for hypersingular integrals and integral equations. J. Sci. Comput. **54**, 145–176 (2013)
10. Sidi, A.: Unified compact numerical quadrature formulas for Hadamard finite parts of singular integrals of periodic functions. Calcolo **58**, 22 (2021)
11. Sidi, A.: Exactness and convergence properties of some recent numerical quadrature formulas for supersingular integrals of periodic functions. Calcolo **58**, 36 (2021)

12. Sidi, A.: PVTSI$^{(m)}$: a novel approach to computation of Hadamard finite parts of nonperiodic singular integrals. Calcolo **59**, 7 (2022)