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Spectrally accurate numerical quadrature formulas for a class of periodic Hadamard Finite Part integrals by regularization

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ABSTRACT

We consider the numerical computation of Hadamard Finite Part (HFP) integrals

$$K_m(t;u) = \oint_0^T S_m\left(\frac{\pi(x-t)}{T}\right)u(x)\,dx, \quad 0<t<T, \quad m \in \{1,2,\dots\},$$

where $u(x)$ is T-periodic and sufficiently differentiable and

$$S_{2r-1}(y) = \frac{\cos y}{\sin^{2r-1} y}, \quad S_{2r}(y) = \frac{1}{\sin^{2r} y}, \quad r=1,2,3,\dots.$$

For each m, we regularize the HFP integral $K_m(t;u)$ and show that

$$K_m(t;u) = K_0(t;U_m) \equiv \int_0^T \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right)U_m(x)\,dx,$$

$U_m(x)$ being some linear combination of the first m derivatives of $u(x)$. We then propose to approximate $K_m(t;u)$ by the quadrature formula $Q_{m,n}(t;u) \equiv K_m(t;\phi_n)$, where $\phi_n(x)$ is the n^{th}-order balanced trigonometric polynomial that interpolates $u(x)$ on $[0,T]$ at the $2n$ equidistant points $x_{n,k} = \frac{kT}{2n}$, $k=0,1,\dots,2n-1$. The implementation of $Q_{m,n}(t;u)$ is simple, the only input needed for this being the $2n$ function values $u(x_{n,k})$, $k = 0,1,\dots,2n-1$. Using Fourier analysis techniques, we develop a complete convergence theory for $Q_{m,n}(t;u)$ as $n\to\infty$ and prove that it enjoys spectral convergence when $u \in C^\infty(\mathbb{R})$. We illustrate the effectiveness of $Q_{m,n}(t;u)$ with numerical examples for $m=0,1,\dots,5$.

We also show that the HFP integral $\oint_0^T f(x,t)\,dx$ of any T-periodic integrand $f(x,t)$ that has m^{th} order poles at $x=t+kT$, $k=0,\pm1,\pm2,\dots$, but is sufficiently differentiable in x on $\mathbb{R}\setminus\{t\pm kT\}_{k=0}^{\infty}$, can be expressed in terms of the $K_s(t;u(\cdot,t))$, where $u(x,t)$ is a T-periodic and sufficiently differentiable function in x on $\mathbb{R}$ that can be computed from $f(x,t)$. Therefore, $\oint_0^T f(x,t)\,dx$ can be computed efficiently using our new numerical quadrature formulas $Q_{s,n}(t;u(\cdot,t))$ on the individual $K_s(t;u(\cdot,t))$. Again, only $2n$ function evaluations, namely, $u(x_{n,k},t)$, $k=0,1,\dots,2n-1$, are needed for the whole process.

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1. Introduction

For an arbitrary function $u(x)$ that is sufficiently differentiable and T-periodic on $\mathbb{R}$, let us define the quantities $K_m(t;u)$, $m = 1, 2, \dots$, via *Hadamard Finite Part (HFP)* integrals as follows:

$$K_m(t;u) = \mathop{\Large{=\!\!\!\!\!\!\int}}_0^T \frac{\cos\frac{\pi(x-t)}{T}}{\sin^m\frac{\pi(x-t)}{T}} u(x)\,dx, \qquad 0 < t < T, \quad \text{if } m = 1, 3, 5, \dots, \tag{1.1a}$$

$$K_m(t;u) = \mathop{\Large{=\!\!\!\!\!\!\int}}_0^T \frac{1}{\sin^m\frac{\pi(x-t)}{T}} u(x)\,dx, \qquad 0 < t < T, \quad \text{if } m = 2, 4, 6, \dots. \tag{1.1b}$$

Clearly, the integrals involved are not defined in the regular sense since their integrands have polar singularities of the form $(x-t)^{-m}$ on $(0, T)$, which are not integrable. They *are* defined in the sense of Hadamard Finite Part (HFP), however.

Let us also define the quantity $K_0(t;u)$ as the *regular* integral

$$K_0(t;u) = \int_0^T \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right) u(x)\,dx, \quad 0 < t < T, \tag{1.2}$$

again with $u(x)$ sufficiently differentiable and T-periodic on $\mathbb{R}$. Observe that the function $(\log|\sin\frac{\pi(x-t)}{T}|)$ is absolutely integrable on $[0, T]$.

Note that the kernel functions in (1.1a), (1.1b), (1.2), namely,

$$\frac{\cos\frac{\pi(x-t)}{T}}{\sin^m\frac{\pi(x-t)}{T}}, \ m = 1, 3, 5, \dots; \qquad \frac{1}{\sin^m\frac{\pi(x-t)}{T}}, \ m = 2, 4, 6, \dots; \quad \log\left|\sin\frac{\pi(x-t)}{T}\right|,$$

are all T-periodic.

In this work, we study the analytical properties of $K_m(t;u)$ *for all* $m \geq 0$ and derive simple and efficient numerical quadrature formulas for them. After presenting some technical preliminaries in Section 2, which are crucial for later developments, we approach the analysis and the approximation of the $K_m(t;u)$ in Sections 3–5 in two major steps:

- First, we *regularize* the HFP integrals in (1.1a)–(1.1b) and show that, for every $m \geq 1$, $K_m(t;u)$ can be expressed as a *regular* integral as

$$K_m(t;u) = K_0(t;U_m) = \int_0^T \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right) U_m(x)\,dx, \quad 0 < t < T. \tag{1.3}$$

  We show that $U_m(x)$ is a linear combination of the first m derivatives of $u(x)$ and is also T-periodic. We provide $U_m(x)$ explicitly.
- Next, we interpolate $u(x)$ at $2n$ equidistant points in $[0, T]$ by a balanced trigonometric polynomial $\phi_n(x)$ and take $K_m(t;\phi_n)$ as our approximation to $K_m(t;u)$. Thus, the cost of computing $K_m(t;\phi_n)$ is only $2n$ evaluations of $u(x)$, no derivative information being needed. For $u \in C^P(\mathbb{R})$, $P > m$, we show by using Fourier analysis techniques that $\lim_{n\to\infty} K_m(t;\phi_n) = K_m(t;u)$. We also provide the rate of convergence and prove that the accuracy of $K_m(t;\phi_n)$ as an approximation to $K_m(t;u)$ increases as P increases. This accuracy is spectral when $P = \infty$ thus when $u \in C^\infty(\mathbb{R})$.

In Section 6, we apply our numerical quadrature formulas to the integrals $K_m(t;u)$, $m = 0, 1, \dots, 5$, with $u(z)$ analytic in an infinite strip containing the real axis; these examples confirm some of our convergence theory pertaining to our numerical quadrature formulas, hence illustrate the strength of our approach. In Section 7, we deal with arbitrary HFP integrals $=\!\!\!\!\!\int_0^T f(x,t)\,dx$, where $f(x,t) = g(x,t)/(x-t)^m$ with $0 < t < T$, and is T-periodic and sufficiently differentiable on $\mathbb{R} \setminus \{t \pm kT\}_{k=0}^\infty$. We show that these integrals can be expressed in terms of the $K_s(t;\cdot)$, hence they can be computed easily with our new numerical quadrature formulas. We also show how these quadrature formulas can be applied to some singular Fredholm integral equations that arise from boundary integral equations. In Section 8, we treat similarly the regular integrals $\int_0^T f(x,t)\,dx$, where $f(x,t)$ is T-periodic on $\mathbb{R}$ and has logarithmic singularities at $t \pm kT$, $k = 0, 1, 2, \dots$. Finally, we include an appendix that gives a short description of trigonometric interpolation and treats the convergence theory for the interpolants and their derivatives as well. This appendix forms an integral part of this work as its contents are used in the development of our numerical quadrature formulas and their convergence analysis.

Note that HFP integrals involving $K_s(t;\cdot)$ arise in a natural way, for example, when dealing with Cauchy transforms of all orders on the unit circle; see [24]. Let us denote by $J_m(z;w)$ the Cauchy transform of order m of the function $w(\zeta)$ on the unit circle, namely,

$$J_m(z; w) = \oint_{\Gamma} \frac{w(\zeta)}{(\zeta - z)^m} \, d\zeta, \quad z \in \Gamma = \{\zeta : |\zeta| = 1\}, \quad m = 1, 2, \dots,$$

Γ being positively oriented. Making the substitution $\zeta = e^{\mathrm{i}x}$, $0 \le x \le 2\pi$, so that $T = 2\pi$, and noting that $z = e^{\mathrm{i}t}$ for some unique $t \in [0, 2\pi)$, and denoting $\widehat{w}(x) = w(e^{\mathrm{i}x})$, $J_m(z; w)$ can be expressed as

$$J_m(z; w) = \frac{\mathrm{i}e^{\mathrm{i}(1-m)t}}{(2\mathrm{i})^m} \oint_0^{2\pi} \frac{\exp\left[\mathrm{i}(2-m)\frac{x-t}{2}\right]}{\sin^m \frac{x-t}{2}} \widehat{w}(x) \, dx.$$

We then have

$$J_1(z; w) = \frac{1}{2}\left[K_1(t; \widehat{w}) + \mathrm{i}\int_0^{2\pi} \widehat{w}(x) \, dx\right],$$

$$J_2(z; w) = -\frac{\mathrm{i}e^{-\mathrm{i}t}}{4} K_2(t; \widehat{w}), \quad J_3(z; w) = -\frac{e^{-\mathrm{i}2t}}{8}\left[K_3(t; \widehat{w}) - \mathrm{i}K_2(t; \widehat{w})\right],$$

$$J_4(z; w) = \frac{\mathrm{i}e^{-\mathrm{i}3t}}{16}\left[K_4(t; \widehat{w}) - 2\mathrm{i}K_3(t; \widehat{w}) - 2K_2(t; \widehat{w})\right].$$

For all $m \ge 2$, we have

$$J_m(z; w) = \frac{\mathrm{i}e^{\mathrm{i}(1-m)t}}{(2\mathrm{i})^m} \sum_{s=2}^{m} \sigma_{m,s} K_s(t; \widehat{w}), \quad \sigma_{m,s} \text{ constants independent of } t.$$

For HFP integrals, we refer the reader to the books by Davis and Rabinowitz [4], Evans [5], Krommer and Ueberhuber [15], and Kythe and Schäferkotter [17]. See also the paper [20] by Monegato for a review. For trigonometric interpolation as summarized in the appendix, see the books by Atkinson [2], Henrici [10], and Zygmund [30], for example. The cases of $m = 0, 1, 2$ appear in the boundary integral formulation of two-dimensional boundary value problems and have been treated by different authors; see Atkinson [1], [3], Kress [13], Kress and Sloan [14], and Kussmaul [16], McLean [18], McLean, Prössdorf, and Wendland [19], Saranen and Sloan [21], Sloan and Burn [28], and Yan and Sloan [29], for example.

Before we go on, we would like to point out to some recent numerical quadrature formulas we developed for computing the HFP integrals

$$\oint_0^T f(x) \, dx, \quad f(x) = \frac{g(x)}{(x-t)^m}, \quad m \in \{1, 2, \dots\}, \quad 0 < t < T,$$

$$f \in C^P(\mathbb{R} \setminus \{t \pm kT\}_{k=0}^{\infty}) \quad \text{and} \quad f(x) \;\; T\text{-periodic}, \tag{1.4}$$

where $P > m$ can be a finite integer or it can be infinite. These formulas are based on the trapezoidal sums

$$h \sum_{j=1}^{n-1} f(t + jh), \quad h = \frac{T}{n}, \quad n = 1, 2, \dots,$$

with compact, yet sophisticated, correction terms involving some or all of $g^{(i)}(t)$, $i = 0, 1, \dots, m$. They are obtained via the application of a recent generalization of the Euler–Maclaurin expansion by the author [22]. They enjoy spectral accuracy when $P = \infty$, just like the methods of the present work. For details about these formulas and numerical examples, see Sidi [23], [24], [25], [27]. For the efficient treatment of nonperiodic $f(x)$, see Sidi [26].

2. Technical preliminaries

We start with some technical tools that we will be using throughout this work, starting already in the next section.

1. First, let us recall that if a function $w(x)$ has a nonintegrable singularity at $x = t$ for $t \in (a, b)$ but is integrable on any subinterval of $[a, b]$ that does not contain $x = t$, then $\oint_a^b w(x) \, dx$, the HFP of $\int_a^b w(x) \, dx$, is obtained by expanding

$$\Lambda(\epsilon) = \int_a^{t-\epsilon} w(x) \, dx + \int_{t+\epsilon}^{b} w(x) \, dx, \quad \epsilon > 0,$$

asymptotically as $\epsilon \to 0+$, discarding those terms that go to infinity, and retaining the limit as $\epsilon \to 0+$ of the remaining terms. (See Monegato [20], for example.)

2. Next, let us define the functions $S_m(y)$ as follows:

$$S_0(y) = \log|\sin y|; \quad S_m(y) = \begin{cases} \dfrac{\cos y}{\sin^m y} & \text{if } m = 1, 3, 5, \dots, \\ \dfrac{1}{\sin^m y} & \text{if } m = 2, 4, 6, \dots. \end{cases} \tag{2.1}$$

Let us now define the functions $R_m(y)$, $m = 1, 2, \dots$, for $y \in (0, \pi)$, via the regular integrals

$$R_m(y) = \int_{\pi/2}^{y} S_m(z)\, dz + R_m(\pi/2), \tag{2.2a}$$

$$R_1(\pi/2) = 0; \quad R_{2r}(\pi/2) = 0, \quad R_{2r+1}(\pi/2) = -\frac{1}{2r}, \quad r = 1, 2, \dots. \tag{2.2b}$$

Then,

$$R_1(y) = S_0(y); \quad R_{2r+1}(y) = -\frac{1}{2r} S_{2r}(y), \quad r = 1, 2, \dots, \tag{2.3}$$

and, by Gradshteyn and Ryzhik [8, §2.515(1)],

$$R_{2r}(y) = -\sum_{k=1}^{r} \alpha_{r,k} S_{2k-1}(y), \quad r = 1, 2, \dots, \tag{2.4}$$

$$\alpha_{r,k} = \begin{cases} \dfrac{1}{2r-1} \dfrac{(2k)(2k+2)\cdots(2r-2)}{(2k-1)(2k+1)\cdots(2r-3)}, & k = 1, 2, \dots, r-1, \\ \dfrac{1}{2r-1}, & k = r. \end{cases} \tag{2.5}$$

Thus

$$\begin{aligned} &m = 2 \to r = 1: \quad && \alpha_{1,1} = 1 \\ &m = 4 \to r = 2: \quad && \alpha_{2,1} = 2/3 \quad \alpha_{2,2} = 1/3 \\ &m = 6 \to r = 3: \quad && \alpha_{3,1} = 8/15 \quad \alpha_{3,2} = 4/15 \quad \alpha_{3,3} = 1/5 \end{aligned}$$

for example. Note that $\sum_{k=1}^{r} \alpha_{r,k} = 1$, which can be proved by induction on r. Clearly, $R_{2r}(y)$ is a linear combination of $S_1(y)$, $S_3(y)$, ..., $S_{2r-1}(y)$, while $R_{2r+1}(y)$ is a constant multiple of $S_{2r}(y)$.
For the special cases $m = 2, 3, 4, 5$, we thus have

$$R_2(y) = -S_1(y), \quad R_3(y) = -\frac{1}{2} S_2(y),$$
$$R_4(y) = -\frac{2}{3} S_1(y) - \frac{1}{3} S_3(y), \quad R_5(y) = -\frac{1}{4} S_4(y).$$

3. Note that, by (2.1), $S_m(y)$ is an even (odd) function of y if m is even (odd), and π-periodic as well. In addition, for $m \geq 1$,

$$S_m(y) = y^{-m} \widehat{S}_m(y), \quad \widehat{S}_m(y) \quad \text{even}, \quad \widehat{S}_m(0) = 1. \tag{2.6}$$

Consequently, by (2.3)–(2.4), $R_m(y)$ is an even (odd) function of y if m is odd (even), and π-periodic as well. In addition, for $m \geq 2$,

$$R_m(y) = y^{-m+1} \widehat{R}_m(y), \quad \widehat{R}_m(y) \quad \text{even}, \quad \widehat{R}_m(0) = -\frac{1}{m-1}. \tag{2.7}$$

4. Finally, we derive a 3-term recursion relation pertaining to the $R_m(y)$ and the $S_m(y)$. We begin with (2.2a)–(2.2b) by applying integration by parts to

$$R_{2r+2}(y) = \int_{\pi/2}^{y} \frac{1}{\sin^{2r} z} \cdot \frac{1}{\sin^2 z}\, dz = \int_{\pi/2}^{y} \left(\frac{1}{\sin^{2r} z} \right) \cdot \frac{d}{dz} \left(-\frac{\cos z}{\sin z} \right) dz.$$

We obtain

$$R_{2r+2}(y) = -\frac{\cos y}{\sin^{2r+1} y} - 2r \int_{\pi/2}^{y} \frac{\cos^2 z}{\sin^{2r+2} z}\, dz,$$

which, upon invoking $\cos^2 z = 1 - \sin^2 z$ and (2.1) and (2.2a)–(2.2b), becomes

$$R_{2r+2}(y) = -S_{2r+1}(y) - 2r[R_{2r+2}(y) - R_{2r}(y)],$$

hence

$$(2r+1)R_{2r+2}(y) = 2rR_{2r}(y) - S_{2r+1}(y). \tag{2.8}$$

Note that (2.3), (2.4), and (2.8) will be very useful later in proving Theorems 3.1 and 4.1.

3. Regularization of $K_m(t;u)$

Let us observe that, by (1.1a)–(1.1b) and (1.2) and (2.1),

$$K_m(t;u) = \fint_0^T S_m\left(\frac{\pi(x-t)}{T}\right) u(x)\, dx. \tag{3.1}$$

Theorems 3.1, 3.2, and 3.3 that follow deal with the issue of regularizing the HFP integrals $K_m(t;u)$ and turn out to be the key to the developments in the next two sections.

Theorem 3.1. *Provided $u \in C^m(\mathbb{R})$ and is T-periodic, and $m \geq 1$, there holds*

$$K_m(t;u) = -\frac{T}{\pi} \fint_0^T R_m\left(\frac{\pi(x-t)}{T}\right) u'(x)\, dx. \tag{3.2}$$

Consequently, by (2.1)–(2.5), we have

$$K_1(t;u) = -\frac{T}{\pi} K_0(t;u'), \tag{3.3}$$

$$K_{2r+1}(t;u) = \frac{T}{\pi}\frac{1}{2r} K_{2r}(t;u'), \quad r = 1, 2, \ldots, \tag{3.4}$$

$$K_{2r}(t;u) = \frac{T}{\pi} \sum_{k=1}^{r} \alpha_{r,k} K_{2k-1}(t;u'), \quad r = 1, 2, \ldots, \tag{3.5}$$

where $\alpha_{r,k}$ are as in (2.4)–(2.5).

Proof. Let us make the change of variable $y = \pi(x-t)/T$ in the integral representation of $K_m(t;u)$ in (3.1). We obtain

$$\begin{aligned} K_m(t;u) &= \frac{T}{\pi} \fint_{-\pi t/T}^{\pi - \pi t/T} S_m(y) w(y)\, dy, \quad w(y) \equiv u\left(t + \frac{T}{\pi} y\right) \\ &= \frac{T}{\pi} \fint_{-\pi/2}^{\pi/2} S_m(y) w(y)\, dy, \end{aligned} \tag{3.6}$$

since the integrand $S_m(y)w(y)$ is π-periodic because both $S_m(y)$ and $w(y)$ are π-periodic.[1]

[1] The HFP integral $\fint_0^T g(x)/(x-t)^m\, dx$, $0 < t < T$, is invariant under a *linear* variable transformation such as $y = \pi(x-t)/T$. This has been known for a long time; see Krommer and Ueberhuber [15, Theorem 1.4.3], for example. In Sidi [26, Theorem 3.1] it is shown that $\fint_0^T g(x)/(x-t)^m\, dx$, $0 < t < T$, is invariant under *arbitrary* legitimate variable transformations.

Noting that $S_m(y) = R'_m(y)$ from (2.2a), let us now apply integration by parts to $\fint_{-\pi/2}^{\pi/2} S_m(y)w(y)\,dy$. We obtain

$$K_m(t;u) = \frac{T}{\pi}\fint_{-\pi/2}^{\pi/2}\left(\frac{d}{dy}[R_m(y)w(y)] - R_m(y)w'(y)\right)dy$$
$$= \frac{T}{\pi}\fint_{-\pi/2}^{\pi/2}\frac{d}{dy}[R_m(y)w(y)]\,dy - \frac{T}{\pi}\fint_{-\pi/2}^{\pi/2} R_m(y)w'(y)\,dy. \tag{3.7}$$

Next, let us show that

$$\fint_{-\pi/2}^{\pi/2}\frac{d}{dy}[R_m(y)w(y)]\,dy = 0. \tag{3.8}$$

Since $\frac{d}{dy}[R_m(y)w(y)]$ is singular only at $y=0$ in the interval $(-\pi/2,\pi/2)$, we have to analyze

$$\Lambda(\epsilon) = \left[\int_{-\pi/2}^{-\epsilon} + \int_{\epsilon}^{\pi/2}\right]\left(\frac{d}{dy}[R_m(y)w(y)]\right)dy$$
$$= \Big[R_m(\pi/2)w(\pi/2) - R_m(-\pi/2)w(-\pi/2)\Big] + \Big[R_m(-\epsilon)w(-\epsilon) - R_m(\epsilon)w(\epsilon)\Big]$$

as $\epsilon \to 0+$. The first brackets vanish because $R_m(y)w(y)$ is π-periodic. Therefore,

$$\Lambda(\epsilon) = R_m(-\epsilon)w(-\epsilon) - R_m(\epsilon)w(\epsilon).$$

We now invoke (2.1)–(2.7). In particular, we invoke the fact that $R_m(y)$ is an even (odd) function when m is an odd (even) integer.

When $m=1$, by the fact that $R_1(y) = S_0(y) = (\log|\sin y|)$, we have

$$\Lambda(\epsilon) = (\log|\sin\epsilon|)\widehat{w}(\epsilon), \quad \widehat{w}(\epsilon) = w(-\epsilon) - w(\epsilon).$$

Since $\widehat{w}(\epsilon)$ is differentiable in a neighborhood of $\epsilon = 0$, we also have that $\widehat{w}(\epsilon) = \widehat{w}'(\widehat{\epsilon})\epsilon$ for some $\widehat{\epsilon} \in (0,\epsilon)$. Therefore, $\lim_{\epsilon\to 0+}\Lambda(\epsilon) = 0$ since $\widehat{w}'(0)$ is well defined. Therefore, (3.8) holds in this case.

When $m \geq 2$, we have the following two cases:

- When m is an even integer, say $m = 2r$, $R_m(\epsilon)$ is an odd function of ϵ, which implies that $R_m(-\epsilon) = -R_m(\epsilon)$, hence

  $$\Lambda(\epsilon) = R_m(\epsilon)\widehat{w}(\epsilon), \quad \widehat{w}(\epsilon) = -w(\epsilon) - w(-\epsilon).$$

  We now express $R_m(\epsilon)$ in the form $R_m(\epsilon) = \epsilon^{-2r+1}\widehat{R}_m(\epsilon)$, with $\widehat{R}_m(\epsilon)$ being an even function of ϵ and infinitely differentiable in a neighborhood of $\epsilon = 0$. Next, we note that $\widehat{w}(\epsilon)$ is an even function of ϵ and m times differentiable in a neighborhood of $\epsilon = 0$. Therefore,

  $$\Lambda(\epsilon) = \epsilon^{-2r+1}\theta(\epsilon), \quad \theta(\epsilon) = \widehat{R}_m(\epsilon)\widehat{w}(\epsilon),$$

  $\theta(\epsilon)$ being even and $m = 2r$ times differentiable in a neighborhood of $\epsilon = 0$. Replacing $\theta(\epsilon)$ by its Maclaurin series expansion with remainder, we obtain

  $$\Lambda(\epsilon) = \epsilon^{-2r+1}\left[\sum_{i=0}^{r-1}\frac{\theta^{(2i)}(0)}{(2i)!}\epsilon^{2i} + \frac{\theta^{(2r)}(\widehat{\epsilon})}{(2r)!}\epsilon^{2r}\right], \quad \text{for some } \widehat{\epsilon} \in (0,\epsilon).$$

  As $\epsilon \to 0+$, the summation $\sum_{i=0}^{r-1}$ contributes to $\Lambda(\epsilon)$ only the (negative) powers ϵ^{-p}, $p = 1,3,\ldots,2r-1$, which we discard. The remaining term, namely, $\frac{\theta^{(2r)}(\widehat{\epsilon})}{(2r)!}\epsilon$, tends to zero as $\epsilon \to 0+$ since $\lim_{\epsilon\to 0+}\theta^{(2r)}(\widehat{\epsilon}) = \theta^{(2r)}(0)$, which is well defined. We have proved the validity of (3.8) when m is even.
- When m is an odd integer, say $m = 2r+1$, $R_m(\epsilon)$ is an even function of ϵ, which implies that $R_m(-\epsilon) = R_m(\epsilon)$, hence

  $$\Lambda(\epsilon) = R_m(\epsilon)\widehat{w}(\epsilon), \quad \widehat{w}(\epsilon) = -w(\epsilon) + w(-\epsilon).$$

  We now express $R_m(\epsilon)$ in the form $R_m(\epsilon) = \epsilon^{-2r}\widehat{R}_m(\epsilon)$, with $\widehat{R}_m(\epsilon)$ being an even function of ϵ and infinitely differentiable in a neighborhood of $\epsilon = 0$. Next, we note that $\widehat{w}(\epsilon)$ is an odd function of ϵ and m times differentiable in a neighborhood of $\epsilon = 0$. Therefore,

$$\Lambda(\epsilon) = \epsilon^{-2r}\theta(\epsilon), \quad \theta(\epsilon) = \widehat{R}_m(\epsilon)\widehat{w}(\epsilon),$$

$\theta(\epsilon)$ being odd and $m = 2r + 1$ times differentiable in a neighborhood of $\epsilon = 0$. Replacing $\theta(\epsilon)$ by its Maclaurin series expansion with remainder, we obtain

$$\Lambda(\epsilon) = \epsilon^{-2r}\left[\sum_{i=0}^{r-1} \frac{\theta^{(2i+1)}(0)}{(2i+1)!}\epsilon^{2i+1} + \frac{\theta^{(2r+1)}(\widehat{\epsilon})}{(2r+1)!}\epsilon^{2r+1}\right] \quad \text{for some } \widehat{\epsilon} \in (0, \epsilon).$$

As $\epsilon \to 0+$, the summation $\sum_{i=0}^{r-1}$ contributes to $\Lambda(\epsilon)$ only the (negative) powers ϵ^{-p}, $p = 1, 3, \dots, 2r - 1$, which we discard. The remaining term, namely, $\frac{\theta^{(2r+1)}(\widehat{\epsilon})}{(2r+1)!}\epsilon$, tends to zero as $\epsilon \to 0+$ since $\lim_{\epsilon\to 0+} \theta^{(2r+1)}(\widehat{\epsilon}) = \theta^{(2r+1)}(0)$, which is well defined. We have proved the validity of (3.8) when m is odd.

We have thus proved the validity of (3.8) for all m. With this, (3.7) becomes

$$K_m(t; u) = -\frac{T}{\pi} \;\;⨎_{-\pi/2}^{\pi/2} R_m(y)w'(y)\,dy,$$

which, upon going back to the variable x and invoking the T-periodicity of $R_m(\frac{\pi(x-t)}{T})u'(x)$, gives (3.2). This completes the proof. ∎

Using Theorem 3.1, we now tackle the task of regularizing the HFP integrals described in (1.1a)–(1.1b). The point here is that the two HFP integral representations of $K_m(t; u)$ given in (3.1) and (3.2) differ essentially in the strength of their respective (polar) singularities at $x = t$; the former has a pole of order m, while the latter has a pole of order $m - 1$. This can also be seen in (3.3)–(3.5). Our ultimate aim is to reach the (regular) integral representation

$$\begin{aligned} &K_m(t; u) = \int_0^T \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right) U_m(x)\,dx = K_0(t; U_m), \quad 0 < t < T, \\ &U_m \in C^\infty(\mathbb{R}) \quad \text{and } T\text{-periodic} \end{aligned} \tag{3.9}$$

described in (1.3). This can be achieved by repeated application of (3.3)–(3.5), starting with $K_m(t; u)$ in (3.1), the end result being that $U_m(x)$ is a linear combination of derivatives of $u(x)$. We will do this by introducing the notation

$$\mu = \frac{T}{2\pi} \tag{3.10}$$

for simplicity. We have already seen in (3.3) the following example with $m = 1$:

$$\begin{aligned} K_1(t; u) &= -\frac{T}{\pi}\int_0^T S_0\left(\frac{\pi(x-t)}{T}\right)u'(x)\,dx \\ &= -2\mu K_0(t; u') \quad \Rightarrow \quad U_1 = -2\mu u'. \end{aligned} \tag{3.11}$$

Here are additional examples with $m \geq 2$:

$$\begin{aligned} K_2(t; u) &= 2\mu K_1(t; u') \\ &= -4\mu^2 K_0(t; u'') \quad \Rightarrow \quad U_2 = -4\mu^2 u'' \end{aligned} \tag{3.12}$$

$$\begin{aligned} K_3(t; u) &= \mu K_2(t; u') \\ &= -4\mu^3 K_0(t; u''') \quad \Rightarrow \quad U_3 = -4\mu^3 u''' \end{aligned} \tag{3.13}$$

$$\begin{aligned} K_4(t; u) &= 2\mu\left[\frac{2}{3}K_1(t; u') + \frac{1}{3}K_3(t; u')\right] \\ &= 2\mu\left[-\frac{4\mu}{3}K_0(t; u'') - \frac{4\mu^3}{3}K_0(t; u^{(4)})\right] \\ &\quad \Rightarrow \quad U_4 = -\frac{8}{3}\mu^2 u'' - \frac{8}{3}\mu^4 u^{(4)} \end{aligned} \tag{3.14}$$

$$\begin{aligned}K_5(t;u) &= \frac{\mu}{2}K_4(t;u') \\ &= \mu^2\Big[-\frac{4\mu}{3}K_0(t;u''') - \frac{4\mu^3}{3}K_0(t;u^{(5)})\Big] \\ &\Rightarrow \quad U_5 = -\frac{4}{3}\mu^3u''' - \frac{4}{3}\mu^5u^{(5)} \end{aligned} \tag{3.15}$$

$$\begin{aligned}K_6(t;u) &= 2\mu\Big[\frac{8}{15}K_1(t;u') + \frac{4}{15}K_3(t;u') + \frac{1}{5}K_5(t;u')\Big] \\ &= 2\mu\Big[-\frac{16\mu}{15}K_0(t;u'') - \frac{4\mu^3}{3}K_0(t;u^{(4)}) - \frac{4\mu^5}{15}K_0(t;u^{(6)})\Big] \\ &\Rightarrow \quad U_6 = -\frac{32}{15}\mu^2u'' - \frac{8}{3}\mu^4u^{(4)} - \frac{8}{15}\mu^6u^{(6)} \end{aligned} \tag{3.16}$$

We summarize the cases of $m \geq 2$ in the next theorem that can be proved by induction on m. [Note that, for $m = 1$, we already have $U_1(x) = -2\mu u'(x)$ by (3.11).]

Theorem 3.2. *For $m \geq 2$, provided $u \in C^m(\mathbb{R})$ and is T-periodic, $U_m(x)$ in (3.9) is some linear combination of $u^{(s)}(x)$, $s = 2, \ldots, m$. Actually, we have*

$$U_m(x) = \begin{cases} \displaystyle\sum_{k=1}^{r} \beta_{r,k}\mu^{2k}u^{(2k)}(x) & \text{if } m = 2r, \quad r = 1, 2, \ldots, \\ \displaystyle\sum_{k=1}^{r} \frac{\beta_{r,k}}{r}\mu^{2k+1}u^{(2k+1)}(x) & \text{if } m = 2r+1, \quad r = 1, 2, \ldots, \end{cases} \tag{3.17}$$

the $\beta_{r,k}$ being fixed (negative) numbers that can be determined by repeated application of Theorem 3.1 to $K_m(t;u)$ in (3.1).[2]

We end this section with the following theorem.

Theorem 3.3. *Provided $u \in C^{2r+2}(\mathbb{R})$ and is T-periodic, the following recursion relation is true:*

$$\begin{aligned}(2r+1)K_{2r+2}(t;u) &= 2rK_{2r}(t;u) + \frac{T}{\pi}K_{2r+1}(t;u') \\ &= 2rK_{2r}(t;u) + \frac{2\mu^2}{r}K_{2r}(t;u''). \end{aligned} \tag{3.18}$$

Proof. By (2.8), we first have

$$\begin{aligned}(2r+1) \fint_0^T R_{2r+2}\Big(\frac{\pi(x-t)}{T}\Big)u'(x)\,dx =& 2r \fint_0^T R_{2r}\Big(\frac{\pi(x-t)}{T}\Big)u'(x)\,dx \\ &- \fint_0^T S_{2r+1}\Big(\frac{\pi(x-t)}{T}\Big)u'(x)\,dx.\end{aligned}$$

By invoking (3.1), (3.2), and (3.4), we obtain (3.18). ■

4. Construction of $K_m(t;u)$ via Fourier series

4.1. Preliminaries

Going back to (1.1a), (1.1b), and (1.2), we realize that, provided $u(x)$ is sufficiently differentiable on $\mathbb{R}$, $K_m(t;u)$ are T-periodic functions of t. This prompts us to study their Fourier series $\sum_{q=-\infty}^{\infty} h_{m,q}e_q(t)$ in the interval $[0, T]$, where

$$e_q(x) \equiv \exp(\mathrm{i}2q\pi x/T), \quad q = 0, \pm1, \pm2, \ldots. \tag{4.1}$$

[2] Explicit expressions for the $\beta_{r,k}$ are given in (4.12)–(4.13) in Theorem 4.2.

As will become clear shortly, working with the functions $\exp(\mathrm{i}2q\pi x/T)$ is much more convenient than working with $\sin(2q\pi x/T)$ and $\cos(2q\pi x/T)$.

We begin with the following important observation concerning $K_m(t; e_q)$:

Theorem 4.1. *For all $m \geq 0$ and all q, there exist constants $L_{m,q}$ independent of t, such that*

$$K_m(t; e_q) = L_{m,q} e_q(t), \quad L_{m,q} = \frac{T}{\pi} \fint_{-\pi/2}^{\pi/2} S_m(y)\, e^{\mathrm{i}2qy}\, dy. \tag{4.2}$$

For $m = 0, 1, 2, 3$, we have

$$L_{0,q} = \begin{cases} -T \log 2 & \text{if } q = 0, \\ -\dfrac{T}{2|q|} & \text{if } q \neq 0, \end{cases} \; L_{1,q} = \mathrm{i}T \operatorname{sgn}(q),\; L_{2,q} = -2T|q|,\; L_{3,q} = -\mathrm{i}2T \operatorname{sgn}(q) q^2, \tag{4.3}$$

where $\operatorname{sgn}(q)$ *is defined as*

$$\operatorname{sgn}(q) = \begin{cases} +1 & \text{if } q > 0 \\ 0 & \text{if } q = 0 \\ -1 & \text{if } q < 0 \end{cases}.$$

For all $m = 2, 3, \ldots$, the $L_{m,q}$ satisfy

$$L_{2r+1,q} = \mathrm{i}\frac{q}{r} L_{2r,q}, \quad r = 1, 2, \ldots, \tag{4.4}$$

and are given as

$$L_{2r,q} = -\frac{Tr}{(r!)\,(1/2)_r} |q| \prod_{j=1}^{r-1} (j^2 - q^2), \tag{4.5}$$

$$L_{2r+1,q} = -\mathrm{i}\frac{T}{(r!)\,(1/2)_r} \operatorname{sgn}(q) q^2 \prod_{j=1}^{r-1} (j^2 - q^2), \tag{4.6}$$

where $(a)_k$ is the Pochhammer symbol defined as

$$(a)_0 = 1, \quad (a)_k = \prod_{j=0}^{k-1} (a + j), \quad k = 1, 2, \ldots.$$

Proof. We start by observing that $e_q(x) = e_q(x - t) \cdot e_q(t)$, hence

$$K_m(t; e_q) = \left[\fint_0^T S_m\!\left(\frac{\pi(x - t)}{T}\right) e_q(x - t)\, dx \right] e_q(t).$$

We now make the change of variable of integration $y = \pi(x - t)/T$, and proceed as in (3.6). Invoking also the fact that $e_q(x - t) = e^{\mathrm{i}2qy}$, and that $S_m(y)e^{\mathrm{i}2qy}$ is π-periodic, we obtain

$$K_m(t; e_q) = \frac{T}{\pi} \left[\fint_{-\pi/2}^{\pi/2} S_m(y) e^{\mathrm{i}2qy}\, dy \right] e_q(t),$$

from which (4.2) follows.

We begin with the case $m = 0$, noting that

$$L_{0,q} = \frac{T}{\pi} \int_{-\pi/2}^{\pi/2} \big(\log|\sin y|\big)\, e^{\mathrm{i}2qy}\, dy. \tag{4.7}$$

By the fact that $\big(\log|\sin y|\big) \sin 2qy$ is odd, we have

$$\int_{-\pi/2}^{\pi/2} \big(\log|\sin y|\big)\sin 2qy\,dy = 0,$$

and, by the fact that $\big(\log|\sin y|\big)\cos 2qy$ is even, we have (see [8, §4.384(7)])

$$\int_{-\pi/2}^{\pi/2} \big(\log|\sin y|\big)\cos 2qy\,dy = 2\int_{0}^{\pi/2} \big(\log\sin y\big)\cos 2qy\,dy$$
$$= \begin{cases} -\pi\log 2 & \text{if } q=0,\\ -\dfrac{\pi}{2|q|} & \text{if } q\neq 0.\end{cases}$$

Substituting these into (4.7), we obtain the result for $L_{0,q}$ given in (4.3).

To obtain the $L_{1,q}$, $L_{2,q}$, and $L_{3,q}$ we make use of (3.11)–(3.13) and the fact that

$$e_q^{(s)}(x) = (\mathrm{i}2q\pi/T)^s e_q(x) = (\mathrm{i}q/\mu)^s e_q(x), \quad s=1,2,\dots, \tag{4.8}$$

where we have also invoked (3.10).

To obtain $L_{m,q}$, $m=2,4,6,\dots$, we recall Theorem 3.3. Letting $u(x)=e_q(x)$ in (3.18) and invoking (4.8), we first have

$$(2r+1)K_{2r+2}(t;e_q) = 2rK_{2r}(t;e_q) - \frac{2q^2}{r}K_{2r}(t;e_q),$$

which results in the 2-term recursion relation

$$K_{2r+2}(t;e_q) = \frac{r^2-q^2}{r(r+1/2)}K_{2r}(t;e_q),$$

whose solution is

$$K_{2r}(t;e_q) = \frac{r}{2(r!)\,(1/2)_r}\left[\prod_{j=1}^{r-1}(j^2-q^2)\right]K_2(t;e_q), \quad r=1,2,\dots.$$

We obtain (4.5) by invoking $K_2(t;e_q)=L_{2,q}e_q(t)$ with $L_{2,q}$ as in (4.3).

As for $L_{m,q}$, $m=3,5,7,\dots$, we invoke (3.4) and (4.8) and obtain (4.4), which, together with $K_{2r}(t;e_q)=L_{2r,q}e_q(t)$, produces (4.6).

This completes the proof. ∎

Note. The following conclusions can be drawn from (4.3) and (4.5):

$$L_{0,0}\neq 0; \quad L_{m,0}=0 \quad \forall m\geq 1; \quad L_{m,-q} = \begin{cases} L_{m,q}, & m=2,4,6,\dots,\\ -L_{m,q}, & m=1,3,5,\dots,\end{cases} \tag{4.9}$$

$$|L_{0,q}| > |L_{0,q'}| \quad \forall |q|<|q'|; \quad |L_{1,q}| = T \quad \forall q\neq 0, \tag{4.10}$$

$$\text{if } m\geq 2\text{:} \begin{cases} L_{m,q}=0 \quad \forall|q|\leq\lfloor m/2\rfloor-1 \quad\text{and}\quad L_{m,q}\neq 0 \quad \forall|q|\geq\lfloor m/2\rfloor,\\ |L_{m,q}|>|L_{m,q'}| \quad \forall|q|>|q'|\geq\lfloor m/2\rfloor.\end{cases} \tag{4.11}$$

We shall make use of these in Section 5.

The next theorem concerns the scalars $\beta_{r,k}$ that we introduced in Theorem 3.2. It explores their connection with the $L_{m,q}$ of the preceding theorem.

Theorem 4.2.

1. *The constants $\beta_{r,k}$ in Theorem 3.2 are related to the $L_{2r,q}$ in (4.2) as follows:*

$$L_{2r,q} = L_{0,q}\sum_{k=1}^{r}(-1)^k\beta_{r,k}q^{2k} = -\frac{T}{2|q|}\sum_{k=1}^{r}(-1)^k\beta_{r,k}q^{2k}, \quad q\neq 0. \tag{4.12}$$

2. *Define the (positive) scalars* $\gamma_{r,k}$ *via*

$$\prod_{j=1}^{r-1}(j^2-q^2)=\sum_{k=1}^{r}(-1)^{k-1}\gamma_{r,k}q^{2k-2}. \tag{4.13}$$

Then the $\beta_{r,k}$ *are given as*

$$\beta_{r,k}=-\frac{2r}{(r!)\,(1/2)_r}\gamma_{r,k}, \quad k=1,\dots,r. \tag{4.14}$$

Proof. Let $u(x)=e_q(x)$ in Theorem 3.2. Thus, $K_{2r}(t;e_q)=K_0(t;E_{q,2r})$ with

$$E_{q,2r}(x)=\left(\sum_{k=1}^{r}(-1)^k\beta_{r,k}q^{2k}\right)e_q(x). \tag{4.15}$$

Consequently,

$$K_{2r}(t;e_q)=\left(\sum_{k=1}^{r}(-1)^k\beta_{r,k}q^{2k}\right)K_0(t;e_q). \tag{4.16}$$

Invoking now $K_m(t;e_q)=L_{m,q}e_q(t)$ on both sides of this equality, we obtain (4.12).

Equating the expressions for $L_{2r,q}$ given in (4.12) and in (4.5), and noting that they are valid for all integer $q\neq 0$, we obtain (4.14). ■

Note that the $\gamma_{r,k}$ can be computed recursively as follows:

$$\gamma_{r,1}=\prod_{j=1}^{r-1}j^2, \quad \gamma_{r,r}=1, \quad r=2,3,\dots,$$
$$\gamma_{r+1,k}=r^2\gamma_{r,k}+\gamma_{r,k-1}, \quad k=2,3,\dots,r, \quad r=2,3,\dots.$$

4.2. Fourier series for $K_0(t;u^{(s)})$, $s=0,1,\dots$

Here and in the sequel, we assume that $u(x)$ is T-periodic in the Hölder class $C^{M+1,\alpha}(\mathbb{R})$, $0<\alpha\le 1$.[3] Then $u(x)$ has the Fourier series representation

$$u(x)\sim\sum_{q=-\infty}^{\infty}c_q e^{\mathrm{i}2q\pi x/T}, \quad c_q=\frac{1}{T}\int_0^T u(x)e^{-\mathrm{i}2q\pi x/T}\,dx, \tag{4.17}$$

which converges [to $u(x)$] absolutely and uniformly since

$$c_q=O(|q|^{-M-\alpha-1}) \quad \text{as } q\to\pm\infty.$$

Similarly, $u^{(s)}(x)$, $s\ge 1$, has the Fourier series representation[4]

$$u^{(s)}(x)\sim\sum_{q=-\infty}^{\infty}c_q^{(s)}e^{\mathrm{i}2q\pi x/T}, \quad c_q^{(s)}=\frac{1}{T}\int_0^T u^{(s)}(x)e^{-\mathrm{i}2q\pi x/T}\,dx=(\mathrm{i}q/\mu)^s c_q, \tag{4.18}$$

which converges [to $u^{(s)}(x)$] absolutely and uniformly for $s\le M$ since

$$c_q^{(s)}=O(|q|^{-M-\alpha-1+s}) \quad \text{as } q\to\pm\infty.$$

We now construct $K_0(t;u^{(s)})$ in terms of the Fourier series representation of $u^{(s)}(x)$, $s=0,1,\dots,M$.

We begin with $u^{(0)}(x)=u(x)$. By the fact that $S_0(\frac{\pi(x-t)}{T})$ is absolutely integrable everywhere and because the Fourier series of $u(x)$ converges to $u(x)$ absolutely and uniformly everywhere, there holds

[3] Thus $u^{(s)}(x)$, $s=0,1,\dots,M$, are all continuous and T-periodic in $\mathbb{R}$, and $u^{(M+1)}(x)$ is in the Hölder class $C^{0,\alpha}(0,T)$, that is, $|u^{(M+1)}(x)-u^{(M+1)}(y)|\le C|x-y|^\alpha$ for all $x,y\in[0,T]$ and for some constant $C>0$. Needless to say, with $M=\infty$, this class contains all T-periodic functions in the class $C^\infty(\mathbb{R})$.

[4] Note that, because $u^{(s)}(0)=u^{(s)}(T)$, $s=0,1,\dots,M$, we have $c_0^{(s)}=0$, $s=1,\dots,M$.

$$K_0(t;u)=K_0\left(t;\sum_{q=-\infty}^{\infty}c_qe_q\right)=\sum_{q=-\infty}^{\infty}c_qK_0(t;e_q),$$

which, by (4.2) and (4.3), becomes

$$K_0(t;u)=\sum_{q=-\infty}^{\infty}c_qL_{0,q}e_q(t)=-Tc_0\log 2-\frac{T}{2}\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\frac{c_q}{|q|}e^{\mathrm{i}2q\pi t/T}. \tag{4.19}$$

Recalling also that $c_0^{(s)}=0$ and $c_q^{(s)}=(\mathrm{i}q/\mu)^sc_q$ for $q\neq 0$, with $s=1,2,\dots$, we similarly have

$$K_0(t;u^{(s)})=\sum_{q=-\infty}^{\infty}c_q^{(s)}L_{0,q}e_q(t)=-\frac{T}{2}\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}(\mathrm{i}q/\mu)^s\frac{c_q}{|q|}e^{\mathrm{i}2q\pi t/T}. \tag{4.20}$$

Clearly, the right-hand side of (4.19) is the Fourier series of $K_0(t;u)$. Similarly, the right-hand side of (4.20) is the Fourier series of $K_0(t;u^{(s)})$ for $s=1,2,\dots$.

4.3. Fourier series for $K_m(t;u)$, $m\geq 1$

Following the developments above, we now proceed to the construction of the Fourier series of $K_m(t;u)$. We assume that $u(x)$ is as in the preceding subsection.

Theorem 4.3. *For $1\leq m\leq M$, $K_m(t;u)$ has the following Fourier series representation that converges absolutely and uniformly:*

$$K_m(t;u)=\sum_{q=-\infty}^{\infty}c_qL_{m,q}e_q(t)=\sum_{q=-\infty}^{\infty}c_qL_{m,q}e^{\mathrm{i}2q\pi t/T}. \tag{4.21}$$

Proof. We begin with the case $m=1$. By (3.11) and (4.20), we have

$$K_1(t;u)=-2\mu K_0(t;u')=\mathrm{i}T\sum_{\substack{q=-\infty\\ q\neq 0}}^{\infty}\operatorname{sgn}(q)c_qe_q(t).$$

Invoking now the fact that $L_{1,q}=\mathrm{i}T\operatorname{sgn}(q)$, we obtain (4.21) with $m=1$.

As for the cases $m\geq 2$, we begin with the regularized $K_m(t;u)$ as described in Theorem 3.2 and treat the cases of $m=2r$ and $m=2r+1$ separately.

For $m=2r$, by Theorem 3.2, we have

$$K_{2r}(t;u)=K_0(t;U_{2r}),\quad U_{2r}(x)=\sum_{k=1}^{r}\beta_{r,k}\mu^{2k}u^{(2k)}(x), \tag{4.22}$$

with the $\beta_{r,k}$ as in Theorem 4.2. Therefore,

$$K_{2r}(t;u)=\sum_{k=1}^{r}\beta_{r,k}\mu^{2k}K_0(t;u^{(2k)}), \tag{4.23}$$

which, upon invoking (4.20), becomes

$$\begin{aligned}K_{2r}(t;u)&=\sum_{k=1}^{r}\beta_{r,k}\mu^{2k}\sum_{q=-\infty}^{\infty}c_q(\mathrm{i}q/\mu)^{2k}L_{0,q}e_q(t)\\&=\sum_{q=-\infty}^{\infty}c_q\left(\sum_{k=1}^{r}(-1)^k\beta_{r,k}q^{2k}\right)L_{0,q}e_q(t).\end{aligned}$$

Invoking now (4.12), we obtain (4.21).

For $m=2r+1$, again by Theorem 3.2, we have

$$K_{2r+1}(t;u)=K_0(t;U_{2r+1}),\quad U_{2r+1}(x)=\sum_{k=1}^{r}\frac{\beta_{r,k}}{r}\mu^{2k+1}u^{(2k+1)}(x). \tag{4.24}$$

Therefore,

$$K_{2r+1}(t;u)=\frac{1}{r}\sum_{k=1}^{r}\beta_{r,k}\mu^{2k+1}K_0(t;u^{(2k+1)}), \tag{4.25}$$

which, upon invoking (4.20), becomes

$$\begin{aligned}K_{2r+1}(t;u)&=\frac{1}{r}\sum_{k=1}^{r}\beta_{r,k}\mu^{2k+1}\sum_{q=-\infty}^{\infty}c_q(\mathrm{i}q/\mu)^{2k+1}L_{0,q}e_q(t)\\&=\sum_{q=-\infty}^{\infty}c_q\left(\mathrm{i}\frac{q}{r}\sum_{k=1}^{r}(-1)^k\beta_{r,k}q^{2k}\right)L_{0,q}e_q(t).\end{aligned}$$

Invoking now (4.12) and (4.5), we obtain (4.21).

This completes the proof. ■

Note. One might think that the result in (4.21) (with $m\geq 1$) should follow immediately by simply writing

$$\begin{aligned}K_m(t;u)&=\neq\!\!\!\!\!\int_0^T S_m\left(\frac{\pi(x-t)}{T}\right)\left(\sum_{q=-\infty}^{\infty}c_qe_q(x)\right)dx\\&=\sum_{q=-\infty}^{\infty}c_q\neq\!\!\!\!\!\int_0^T S_m\left(\frac{\pi(x-t)}{T}\right)e_q(x)dx\\&=\sum_{q=-\infty}^{\infty}c_qL_{m,q}e_q(t).\end{aligned}$$

Despite the fact that the (infinite) series $\sum_{q=-\infty}^{\infty}c_qe_q(x)$ converges to $u(x)$ absolutely and uniformly on $\mathbb{R}$, the equality on the second line cannot be justified. The reason for this is that, when $m\geq 1$, the integral on the first line does not exist in the regular sense as its integrand has a nonintegrable singularity at $x=t$ in $(0,T)$.

5. Numerical quadrature formula for $K_m(t;u)$ via trigonometric interpolation

5.1. The numerical quadrature formula $Q_{m,n}(t;u)$

So far, we have seen that the T-periodic (divergent) HFP integrals in (1.1a)–(1.1b) can be expressed in terms of the HFP integrals $K_m(t;u)$, which can be expressed as the *regular* integrals $K_0(t;U_m)=\int_0^T S_0\big(\frac{\pi(x-t)}{T}\big)U_m(x)\,dx$, $U_m(x)$ being a linear combination of derivatives of $u(x)$ as in (3.11)–(3.16) for $m=1,\dots,6$ and as in (3.17) for arbitrary $m\geq 2$. We now present a quadrature method that approximates $K_0(t;U_m)$ without having to approximate the individual $K_0(t;u^{(s)})$ that form $K_0(t;U_m)$.

We proceed as follows:

- We first approximate $u(x)$ on $[0,T]$ by a balanced trigonometric polynomial $\phi_n(x)$ of degree n that interpolates $u(x)$ at $2n$ equidistant points $x_{n,0},x_{n,1},\dots,x_{n,2n-1}$. As summarized in the appendix to this work, $\phi_n(x)$ is of the form

$$\phi_n(x)=\sum_{q=-n}^{n}{}''\tilde{c}_{n,q}e_q(x),\quad \tilde{c}_{n,n}=\tilde{c}_{n,-n}, \tag{5.1}$$

the double prime on the summation $\sum''^{\,n}_{q=-n}$ meaning that the terms with $q=\pm n$ are to be multiplied by 1/2, and

$$\phi_n(x_{n,k})=u(x_{n,k}),\quad x_{n,k}=\frac{kT}{2n},\quad k=0,1,\dots,2n-1, \tag{5.2}$$

and

$$\tilde{c}_{n,q}=\frac{1}{2n}\sum_{k=0}^{2n-1}\overline{e_q(x_{n,k})}u(x_{n,k})=\frac{1}{2n}\sum_{k=0}^{2n-1}e^{-\mathrm{i}qk\pi/n}u(x_{n,k}),\quad -n\leq q\leq n. \tag{5.3}$$

In addition, the $\tilde{c}_{n,q}$ are related to the c_q as in

$$\tilde{c}_{n,p} = c_p + \sum_{\substack{\tau=-\infty \\ \tau\neq 0}}^{\infty} c_{p+2n\tau}. \tag{5.4}$$

- Next, we approximate $K_m(t; u)$ by $K_m(t; \phi_n)$. That is, our numerical quadrature formula $Q_{m,n}(t; u)$ for $K_m(t; u)$ is simply

$$Q_{m,n}(t; u) = K_m(t; \phi_n). \tag{5.5}$$

Thus, because $\phi_n(x) = \sum''^{n}_{q=-n} \tilde{c}_{n,q} e_q(x)$ is a finite sum, we can immediately write

$$Q_{m,n}(t; u) = K_m\left(t; \sum''^{n}_{q=-n} \tilde{c}_{n,q} e_q\right) = \sum''^{n}_{q=-n} \tilde{c}_{n,q} K_m(t; e_q), \tag{5.6}$$

which, upon invoking (4.2), becomes

$$Q_{m,n}(t; u) = \sum''^{n}_{q=-n} \tilde{c}_{n,q} L_{m,q} e_q(t). \tag{5.7}$$

Clearly, in this form, $Q_{m,n}(t; u)$ is very easy to compute once the $\tilde{c}_{n,q}$ have been computed.

Substituting (5.3) into (5.7) and rearranging, we also obtain $Q_{m,n}(t; u)$ as a trigonometric sum as follows:

$$Q_{m,n}(t; u) = \frac{1}{2n} \sum_{k=0}^{2n-1} \left[\sum''^{n}_{q=-n} L_{m,q} e_q(t - x_{n,k}) \right] u(x_{n,k}). \tag{5.8}$$

Remarks 1.

1. By (5.1)–(5.8), it is clear that the only input we need for computing $Q_{m,n}(t; u)$ is the set $\{u(x_{n,0}), u(x_{n,1}), \dots, u(x_{n,2n-1})\}$, which we use for computing the $\tilde{c}_{n,q}$; no derivative information from $u(x)$ is required.
2. By (3.11)–(3.17) and (5.5), we have that

$$Q_{m,n}(t; u) = K_m(t; \phi_n) = K_0(t; \Phi_{m,n}),$$

where

$$\Phi_{m,n} = -2\mu\phi_n' \quad \text{if } m = 1$$

while

$$\Phi_{m,n} = \begin{cases} \sum_{k=1}^{r} \beta_{r,k} \mu^{2k} \phi_n^{(2k)} & \text{if } m = 2r, \quad r = 1, 2, \dots, \\ \sum_{k=1}^{r} \frac{\beta_{r,k}}{r} \mu^{2k+1} \phi_n^{(2k+1)} & \text{if } m = 2r+1, \quad r = 1, 2, \dots. \end{cases}$$

This means that our numerical quadrature formula $Q_{m,n}(t; u)$ replaces $u^{(s)}(x)$ in the composition of $U_m(x)$ resulting from $K_m(t; u) = K_0(t; U_m)$ by $\phi_n^{(s)}(x)$. This takes place only *implicitly*, however, as is obvious from (5.7), since the $L_{m,q}$ are readily available by Theorem 4.1.
3. Even though $\phi_n(x_{n,k}) = u(x_{n,k})$, we have only $\phi_n^{(s)}(x_{n,k}) \approx u^{(s)}(x_{n,k})$, $k = 0, 1, \dots, 2n-1$, for $s = 1, 2, \dots$.

5.2. Convergence of $Q_{m,n}(t; u)$

We now turn to the study of the convergence as $n \to \infty$ of $Q_{m,n}(t; u)$. We begin by deriving upper bounds on the absolute errors $|Q_{m,n}(t; u) - K_m(t; u)|$ that we express in terms of the Fourier coefficients of $u(x)$.

Theorem 5.1. *For each m and n, the absolute errors $|Q_{m,n}(t; u) - K_m(t; u)|$ can be bounded as follows:*

$$\text{for } m = 0: \quad |Q_{0,n}(t; u) - K_0(t; u)| \le \sum''_{|q|\ge n} |c_q| (|L_{0,0}| + |L_{0,q}|), \tag{5.9}$$

$$\text{for } m = 1: \quad |Q_{1,n}(t; u) - K_1(t; u)| \le 2T \sum''_{|q|\ge n} |c_q|, \tag{5.10}$$

$$\text{for } m \ge 2: \quad |Q_{m,n}(t; u) - K_m(t; u)| \le \sum''_{|q|\ge n} |c_q| (|L_{m,n}| + |L_{m,q}|). \tag{5.11}$$

Note that all the bounds here are independent of t.

Proof. By (4.21), (5.4), and (5.7), we have

$$\begin{aligned} Q_{m,n}(t;u)-K_m(t;u) &= \sideset{}{''}\sum_{p=-n}^{n} \tilde{c}_{n,p}L_{m,p}e_p(t) - \sum_{q=-\infty}^{\infty} c_qL_{m,q}e_q(t) \\ &= \sideset{}{''}\sum_{|p|\le n}\Big(c_p+\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\Big)L_{m,p}e_p(t) - \sum_{q=-\infty}^{\infty} c_qL_{m,q}e_q(t) \\ &= \sideset{}{''}\sum_{|p|\le n}\Big(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\Big)L_{m,p}e_p(t) - \sideset{}{''}\sum_{|q|\ge n} c_qL_{m,q}e_q(t), \end{aligned}$$

which, upon taking moduli, gives

$$|Q_{m,n}(t;u)-K_m(t;u)| \le \sideset{}{''}\sum_{|p|\le n}\Big(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Big)|L_{m,p}| + \sideset{}{''}\sum_{|q|\ge n} |c_q||L_{m,q}|. \tag{5.12}$$

To continue, we make use of (4.9)–(4.11) that follow the proof of Theorem 4.1.

For $m=0$, we start by noting that $|L_{0,0}|>|L_{0,\pm1}|>|L_{0,\pm2}|>\cdots$. Therefore, (5.12) gives

$$|Q_{0,n}(t;u)-K_0(t;u)| \le |L_{0,0}|\Big(\sideset{}{''}\sum_{|p|\le n}\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Big) + \sideset{}{''}\sum_{|q|\ge n} |c_q||L_{0,q}|,$$

which, by Lemma A.2, results in (5.9).

For $m=1$, by the fact that $|L_{1,q}|=T$ for all $q\neq 0$, (5.12) becomes

$$|Q_{1,n}(t;u)-K_1(t;u)| \le T\sideset{}{''}\sum_{|p|\le n}\Big(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Big) + T\sideset{}{''}\sum_{|q|\ge n} |c_q|,$$

which, by Lemma A.2, results in (5.10).

For $m=2,3,\dots,$ recalling that for $|q|\ge\lfloor m/2\rfloor$, we have $|L_{m,q}|\ge|L_{m,q'}|$ for $|q|\ge|q'|$, we have

$$\begin{aligned} \sideset{}{''}\sum_{|p|\le n}\Big(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Big)|L_{m,p}| &\le |L_{m,n}|\Big(\sideset{}{''}\sum_{|p|\le n}\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Big) \\ &= |L_{m,n}|\sideset{}{''}\sum_{|q|\ge n}|c_q| \quad \text{by Lemma A.2}, \end{aligned} \tag{5.13}$$

which, upon substituting into (5.12), gives (5.11). ∎

The next theorem provides the rates at which $Q_{m,n}(t;u)$ converges to $K_m(t;u)$ as $n\to\infty$ for all $t\in\mathbb{R}$. It follows from Theorem 5.1 and from

$$c_q = O(|q|^{-M-\alpha-1}) \quad \text{and} \quad L_{m,q}=O(|q|^{m-1}) \quad \text{as } q\to\pm\infty.$$

Theorem 5.2.

1. *If $u(x)$ is T-periodic and in the Hölder class $C^{M+1,\alpha}(0,T)$, and if $m\le M$, then*

$$\text{for } m=0:\quad |Q_{0,n}(t;u)-K_0(t;u)| = O(n^{-M-\alpha}) \quad \text{as } n\to\infty, \tag{5.14}$$

$$\text{for } m\ge 1:\quad |Q_{m,n}(t;u)-K_m(t;u)| = O(n^{-M-\alpha-1+m}) \quad \text{as } n\to\infty. \tag{5.15}$$

2. *If $u(x)$ is T-periodic and in $C^\infty(\mathbb{R})$, then, for $m\ge 0$,*

$$|Q_{m,n}(t;u)-K_m(t;u)| = o(n^{-\lambda}) \quad \text{as } n\to\infty \quad \forall\lambda>0, \tag{5.16}$$

that is, $Q_{m,n}(t;u)$ converges spectrally.

3. *In case $u(z)$ is also T-periodic and analytic in an infinite strip D_ρ of the complex z-plane, where*

$$D_\rho = \{z \in \mathbb{C} : |\boldsymbol{Im}\, z| < \rho\}, \tag{5.17}$$

then, for $m \geq 0$,

$$|Q_{m,n}(t;u) - K_m(t;u)| = O(e^{-2n\pi\theta/T}) \quad \text{as } n \to \infty, \quad \forall \theta \in (0,\rho). \tag{5.18}$$

All these results are valid uniformly in t.

5.3. Exactness property of $Q_{m,n}(t;u)$

We now prove some exactness results pertaining to $Q_{m,n}(t;u)$.

Theorem 5.3. *The numerical quadrature formula $Q_{m,n}(t;u)$ has the following exactness property:*

$$Q_{m,n}(t;u) = K_m(t;u) \quad \text{if } u(x) = {\sum_{q=-n}^{n}}'' c_q e_q(x) \text{ with } c_n = c_{-n}.$$

In particular,

$$Q_{m,n}(t;e_q) = K_m(t;e_q) = L_{m,q}e_q(t), \quad q = 0, \pm 1, \dots, \pm(n-1),$$
$$Q_{m,n}(t;e_n + e_{-n}) = K_m(t;e_n + e_{-n}) = L_{m,n}e_n(t) + L_{m,-n}e_{-n}(t).$$

In words, $Q_{m,n}(t;u)$ reproduces $K_m(t;u)$ when $u(x)$ is a balanced trigonometric polynomial of degree at most n.

Proof. By Lemma A.1 in the appendix when $u(x) = {\sum''}_{q=-n}^{n} c_q e_q(x)$ with $c_n = c_{-n}$, there holds $\phi_n(x) \equiv u(x)$. Therefore,

$$Q_{m,n}(t;u) = K_m(t;\phi_n) = K_m(t;u).$$

This completes the proof. ■

Following Theorem 5.3, which provides $Q_{m,n}(t;e_q)$ for $|q| \leq n-1$, Theorem 5.4 below provides $Q_{m,n}(t;e_q)$ for $|q| \geq n$.

Theorem 5.4. *Define the set of integers Γ as $\Gamma = \{\pm n, \pm 3n, \pm 5n, \dots\}$.*

1. *If $q \in \Gamma$, then $q = (2j+1)n$ for some integer j, and*

$$Q_{m,n}(t;e_q) = Q_{m,n}(t;e_{\pm n}) = \frac{1}{2}[L_{m,n}e_n(t) + L_{m,-n}e_{-n}(t)]. \tag{5.19}$$

2. *If $q \notin \Gamma$, then there exist unique integers τ and s, $|s| \leq n-1$, such that $q = 2n\tau + s$, and*

$$Q_{m,n}(t;e_q) = Q_{m,n}(t;e_s) = L_{m,s}e_s(t). \tag{5.20}$$

Proof. We begin by rewriting (5.8) in the form

$$Q_{m,n}(t;u) = \sum_{k=0}^{2n-1} W_{m,n,k}(t)u(x_{n,k}), \quad W_{m,n,k}(t) = \frac{1}{2n}{\sum_{p=-n}^{n}}'' L_{m,p}e_p(t - x_{n,k}). \tag{5.21}$$

Clearly, $W_{m,n,k}(t)$ are independent of $u(x)$. This implies that if $u_1(x)$ and $u_2(x)$ are two different functions for which $u_1(x_{n,k}) = u_2(x_{n,k})$, $k = 0, 1, \dots, 2n-1$, then $Q_{m,n}(t;u_1) = Q_{m,n}(t;u_2)$. We make use of this fact in the sequel.

Now let $u(x) = e_q(x)$ in (5.21) and note that $e_q(x_{n,k}) = e^{\mathrm{i}qk\pi/n}$. There are two cases to consider:

1. $q \in \Gamma$: In this case, $q = (2j+1)n$ for some integer j, and we have

$$e_q(x_{n,k}) = e^{\mathrm{i}(2j+1)k\pi} = (-1)^k = e_{\pm n}(x_{n,k}) \quad \text{independent of } j.$$

When substituted into (5.21), this gives

$$Q_{m,n}(t;e_q) = Q_{m,n}(t;e_{\pm n}) \quad \forall q \in \Gamma.$$

Now, the balanced trigonometric interpolant of degree n to $u(x) = e_{\pm n}(x)$ is simply

$$\widehat{\phi}_n(x) = \frac{1}{2}[e_n(x) + e_{-n}(x)].$$

The result in (5.19) follows by invoking $Q_{m,n}(t; e_{\pm n}) = K_m(t; \widehat{\phi}_n)$ and Theorem 5.3.

2. $q \notin \Gamma$: In this case, $q = 2n\tau + s$ with unique integers τ and s, $|s| \le n-1$, as can be verified easily. Next,

$$e_q(x_{n,k}) = e^{\mathrm{i}(2\tau k\pi + sk\pi/n)} = e^{\mathrm{i}sk\pi/n} = e_s(x_{n,k}).$$

When substituted in (5.21), this gives

$$Q_{m,n}(t; e_q) = Q_{m,n}(t; e_s).$$

Since $|s| \le n-1$, we can now invoke Theorem 5.3 and obtain (5.20).

This completes the proof. ■

6. Numerical examples

We now apply the numerical method we have just developed to the integrals $K_m(t; u)$, $m = 0, 1, \dots, 5$, with $T = 2\pi$ (hence $\mu = 1$), where

$$u(x) = \sum_{q=0}^{\infty} \eta^q \cos qx = \boldsymbol{Re}\frac{1}{1 - \eta e^{\mathrm{i}x}}, \quad \eta \text{ real}, \quad 0 < \eta < 1. \tag{6.1}$$

Clearly, $u(x)$ is infinitely differentiable and 2π-periodic on $\mathbb{R}$. In addition, $u(x)$ can be continued to the complex z-plane, such that $u(z)$, is also 2π-periodic and analytic in the infinite strip

$$D_\rho = \{z \in \mathbb{C} : |\boldsymbol{Im}\, z| < \rho\},$$

with $\rho = \log \eta^{-1}$. Finally, $K_m(t; u)$ can be computed numerically by summing the Fourier series in (4.21) that converges quickly in our case here. We first have

$$c_0 = 1; \quad c_q = \eta^{|q|}/2, \quad q \neq 0.$$

Next, by (4.19) and (4.21),

$$K_0(t; u) = -2\pi \log 2 - \pi \sum_{q=1}^{\infty} \frac{1}{q}\eta^q \cos qt = \pi \log\left(\frac{|1 - \eta e^{\mathrm{i}t}|}{4}\right)$$

$$K_1(t; u) = -2\pi \sum_{q=1}^{\infty} \eta^q \sin qt = -2\pi\, \boldsymbol{Im}\frac{1}{1 - \eta e^{\mathrm{i}t}}$$

$$K_2(t; u) = -4\pi \sum_{q=1}^{\infty} q\eta^q \cos qt = -4\pi\, \boldsymbol{Re}\frac{\eta e^{\mathrm{i}t}}{(1 - \eta e^{\mathrm{i}t})^2}$$

$$K_3(t; u) = 4\pi \sum_{q=1}^{\infty} q^2\eta^q \sin qt = 4\pi\, \boldsymbol{Im}\frac{\eta e^{\mathrm{i}t}(1 + \eta e^{\mathrm{i}t})}{(1 - \eta e^{\mathrm{i}t})^3}$$

$$K_4(t; u) = \frac{8\pi}{3} \sum_{q=1}^{\infty} q(q^2 - 1)\eta^q \cos qt$$

$$K_5(t; u) = -\frac{4\pi}{3} \sum_{q=1}^{\infty} q^2(q^2 - 1)\eta^q \sin qt$$

and so on.

We have applied our quadrature formulas $Q_{m,n}(t; u)$ as shown in (5.7), to the integrals $K_m(t; u)$ with $t = 1$ using quadruple-precision arithmetic with roundoff unit $\mathbf{u} = 1.93 \times 10^{-34}$. The results of our computations for $m = 0, 1, \dots, 5$ are shown in Tables 6.0–6.5. Note that because $u(z)$ is analytic in the infinite strip D_ρ with $\rho = \log \eta^{-1}$, we have that the error $[Q_{m,n}(t; u) - K_m(t; u)]$ tends to zero as $n \to \infty$ exponentially in n like η^n by Theorem 5.2. Our numerical results confirm this amply for the different values of η. Finally, we recall that $Q_{m,n}(t; u)$ requires only $2n$ evaluations of $u(x)$ and no evaluations of derivatives of $u(x)$.

Table 6.0
Numerical results for $Q_{0,n}(t;u)$ with $t=1$ and $u(x)$ as in (6.1). Here $E_n(\eta=c)=|Q_{0,n}(t;u)-K_0(t;u)|/|K_0(t;u)|$ for $\eta=c$.

n	$E_n(\eta=0.1)$	$E_n(\eta=0.2)$	$E_n(\eta=0.3)$	$E_n(\eta=0.4)$	$E_n(\eta=0.5)$
20	$6.00D-23$	$1.33D-16$	$6.93D-13$	$2.99D-10$	$3.28D-08$
40	$3.41D-34$	$5.62D-31$	$9.84D-24$	$1.36D-18$	$1.31D-14$
60	$6.82D-34$	$3.31D-34$	$3.23D-34$	$3.98D-27$	$3.26D-21$
80	$0.00D+00$	$0.00D+00$	$6.45D-34$	$1.27D-33$	$7.82D-27$
100	$5.12D-34$	$4.96D-34$	$6.45D-34$	$1.59D-34$	$3.15D-33$
120	$3.41D-34$	$0.00D+00$	$6.45D-34$	$3.17D-34$	$3.15D-34$

Table 6.1
Numerical results for $Q_{1,n}(t;u)$ with $t=1$ and $u(x)$ as in (6.1). Here $E_n(\eta=c)=|Q_{1,n}(t;u)-K_1(t;u)|/|K_1(t;u)|$ for $\eta=c$.

n	$E_n(\eta=0.1)$	$E_n(\eta=0.2)$	$E_n(\eta=0.3)$	$E_n(\eta=0.4)$	$E_n(\eta=0.5)$
20	$8.16D-21$	$8.56D-15$	$2.85D-11$	$8.97D-09$	$7.78D-07$
40	$4.80D-32$	$1.47D-28$	$1.62D-21$	$1.61D-16$	$1.21D-12$
60	$5.26D-33$	$7.50D-34$	$8.24D-32$	$2.53D-24$	$1.65D-18$
80	$3.37D-32$	$1.40D-32$	$5.58D-33$	$1.33D-34$	$1.83D-25$
100	$1.21D-31$	$6.17D-32$	$4.15D-32$	$2.82D-32$	$1.34D-30$
120	$2.28D-32$	$9.45D-33$	$4.84D-33$	$6.63D-34$	$1.34D-33$

Table 6.2
Numerical results for $Q_{2,n}(t;u)$ with $t=1$ and $u(x)$ as in (6.1). Here $E_n(\eta=c)=|Q_{2,n}(t;u)-K_2(t;u)|/|K_2(t;u)|$ for $\eta=c$.

n	$E_n(\eta=0.1)$	$E_n(\eta=0.2)$	$E_n(\eta=0.3)$	$E_n(\eta=0.4)$	$E_n(\eta=0.5)$
20	$7.94D-19$	$1.63D-12$	$7.41D-08$	$1.43D-06$	$6.48D-05$
40	$8.67D-31$	$2.82D-26$	$4.22D-18$	$2.53D-14$	$9.85D-11$
60	$1.88D-30$	$1.85D-30$	$7.32D-29$	$1.74D-22$	$6.00D-17$
80	$1.35D-29$	$1.28D-29$	$1.11D-28$	$1.31D-29$	$2.42D-22$
100	$1.25D-29$	$1.40D-29$	$1.43D-28$	$7.07D-30$	$1.49D-28$
120	$9.70D-30$	$6.91D-30$	$4.68D-29$	$1.59D-30$	$5.41D-31$

Table 6.3
Numerical results for $Q_{3,n}(t;u)$ with $t=1$ and $u(x)$ as in (6.1). Here $E_n(\eta=c)=|Q_{3,n}(t;u)-K_3(t;u)|/|K_3(t;u)|$ for $\eta=c$.

n	$E_n(\eta=0.1)$	$E_n(\eta=0.2)$	$E_n(\eta=0.3)$	$E_n(\eta=0.4)$	$E_n(\eta=0.5)$
20	$2.80D-18$	$2.49D-12$	$8.12D-09$	$3.15D-06$	$5.80D-04$
40	$3.06D-29$	$1.60D-25$	$1.86D-18$	$2.45D-13$	$4.24D-09$
60	$1.90D-29$	$1.07D-29$	$1.99D-28$	$8.49D-21$	$1.25D-14$
80	$1.31D-28$	$5.06D-29$	$3.13D-29$	$9.63D-30$	$2.17D-21$
100	$4.11D-28$	$1.88D-28$	$1.33D-28$	$1.27D-28$	$2.88D-26$
120	$2.84D-28$	$1.18D-28$	$6.98D-29$	$5.02D-29$	$5.88D-29$

Table 6.4
Numerical results for $Q_{4,n}(t;u)$ with $t=1$ and $u(x)$ as in (6.1). Here $E_n(\eta=c)=|Q_{4,n}(t;u)-K_4(t;u)|/|K_4(t;u)|$ for $\eta=c$.

n	$E_n(\eta=0.1)$	$E_n(\eta=0.2)$	$E_n(\eta=0.3)$	$E_n(\eta=0.4)$	$E_n(\eta=0.5)$
20	$2.56D-16$	$9.52D-11$	$1.84D-07$	$4.74D-05$	$4.95D-03$
40	$5.46D-28$	$6.68D-24$	$4.16D-17$	$3.26D-12$	$2.86D-08$
60	$1.05D-26$	$1.76D-27$	$1.34D-27$	$5.26D-20$	$4.24D-14$
80	$3.03D-26$	$5.21D-27$	$1.84D-27$	$5.33D-27$	$2.89D-19$
100	$5.48D-26$	$1.20D-26$	$5.43D-27$	$3.63D-27$	$2.70D-25$
120	$3.59D-26$	$2.23D-27$	$2.63D-28$	$7.87D-28$	$1.25D-27$

Table 6.5
Numerical results for $Q_{5,n}(t;u)$ with $t=1$ and $u(x)$ as in (6.1). Here $E_n(\eta=c)=|Q_{5,n}(t;u)-K_5(t;u)|/|K_5(t;u)|$ for $\eta=c$.

n	$E_n(\eta=0.1)$	$E_n(\eta=0.2)$	$E_n(\eta=0.3)$	$E_n(\eta=0.4)$	$E_n(\eta=0.5)$
20	$1.56D-15$	$6.16D-09$	$1.18D-06$	$1.19D-04$	$6.06D-03$
40	$4.23D-26$	$1.47D-21$	$1.08D-15$	$4.03D-11$	$2.12D-07$
60	$1.30D-25$	$3.00D-25$	$2.57D-25$	$3.08D-18$	$1.36D-12$
80	$1.13D-24$	$2.12D-24$	$8.61D-26$	$1.23D-26$	$3.63D-19$
100	$3.37D-24$	$7.24D-24$	$3.23D-25$	$8.71D-26$	$8.81D-24$
120	$3.25D-24$	$6.49D-24$	$2.47D-25$	$5.31D-26$	$2.20D-26$

7. Application to general $\displaystyle\neq\!\!\!\!\!\!\int_0^T f(x,t)\,dx$ with $m \geq 1$

7.1. Treatment of general $f(x,t)$

So far, we have dealt with the HFP integrals $K_m(t;u) = \neq\!\!\!\!\!\!\int_0^T f(x,t)\,dx$, where $f(x,t)$ is T-periodic in x and is expressed as [recall (1.1a)–(1.1b)]

$$f(x,t) = \begin{cases} \dfrac{\cos\frac{\pi(x-t)}{T}}{\sin^m\frac{\pi(x-t)}{T}}u(x)\,dx, & 0<t<T, \quad \text{if } m=1,3,5,\dots, \\ \dfrac{1}{\sin^m\frac{\pi(x-t)}{T}}u(x)\,dx, & 0<t<T, \quad \text{if } m=2,4,6,\dots, \end{cases} \tag{7.1}$$

$u(x)$ being T-periodic and sufficiently differentiable on $\mathbb{R}$. We also mentioned that the Cauchy transforms $J_m(t;w)$ on the unit circle described in Section 1 are actually linear combinations of $K_s(t;\widehat{w})$, $s=1,2,\dots,m$, where $\widehat{w}(x) = w(e^{ix})$.

We now consider the general HFP integrals $I_m(t;f) = \neq\!\!\!\!\!\!\int_0^T f(x,t)\,dx$, where $f(x,t)$ is T-periodic in $x \in \mathbb{R}$ and has the general form

$$f(x,t) = \frac{g(x,t)}{(x-t)^m}, \quad 0<t<T, \quad m\in\{1,2,\dots\}, \tag{7.2}$$

such that $g(x,t)$ is sufficiently differentiable as a function of x on $\mathbb{R}$. In addition, t is being held fixed throughout. [Note that $g(x,t)$ is *not* T-periodic since $(x-t)^{-m}$ is not.]

Let us define the functions $v_m(x,t)$ as follows:

$$v_m(x,t) = \begin{cases} \dfrac{\exp\left(\mathrm{i}\frac{\pi(x-t)}{T}\right)}{\sin^m\frac{\pi(x-t)}{T}} & \text{if } m=1,3,5,\dots, \\ \dfrac{1}{\sin^m\frac{\pi(x-t)}{T}} & \text{if } m=2,4,6,\dots. \end{cases} \tag{7.3}$$

Clearly, for each $m=1,2,\dots$, and with fixed t, $v_m(x,t)$ is T-periodic and also nonzero for $x\in\mathbb{R}$ with poles of order m at $x=t\pm kT$, $k=0,1,2,\dots$, and is infinitely differentiable on $\mathbb{R}\setminus\{t\pm kT\}_{k=0}^{\infty}$. In addition, $v_m(z,t)$ continues to be T-periodic and nonzero in the whole complex z-plane and is meromorphic with poles of order m at $z=t\pm kT$, $k=0,1,2,\dots$.

We can now express $f(x,t)$ in the form

$$f(x,t) = u(x,t)v_m(x,t) \quad \Rightarrow \quad u(x,t) = \frac{f(x,t)}{v_m(x,t)} = \frac{g(x,t)}{(x-t)^m v_m(x,t)}. \tag{7.4}$$

Clearly, $u(x,t)$ is T-periodic and sufficiently differentiable on $\mathbb{R}$ because $v_m(x,t)$ does not vanish anywhere in $\mathbb{R}$.[5] Assuming that $g(x,t)$ is known/computable, we can easily compute $u(x,t)$ numerically from

$$u(x,t) = \left(\frac{\pi}{T}\right)^m \times \begin{cases} g(x,t)\left[\operatorname{sinc}\left(\frac{\pi(x-t)}{T}\right)\right]^m e^{-\mathrm{i}\pi(x-t)/T} & \text{if } m=1,3,5,\dots, \\ g(x,t)\left[\operatorname{sinc}\left(\frac{\pi(x-t)}{T}\right)\right]^m & \text{if } m=2,4,6,\dots, \end{cases} \tag{7.5}$$

where

$$\operatorname{sinc}(z) = \frac{\sin z}{z}$$

is the sinc function, which is defined and is easily computable everywhere and is positive for $z\in(-\pi,\pi)$, with $\operatorname{sinc}(0)=1$. Therefore, we also have that

$$u(t,t) = \left(\frac{\pi}{T}\right)^m g(t,t) \quad \forall m\geq 1.$$

Comparing (7.3) with (2.1), we realize that

$$v_m(x,t) = \begin{cases} S_1\left(\frac{\pi(x-t)}{T}\right) + \mathrm{i} & \text{if } m=1, \\ S_{2r+1}\left(\frac{\pi(x-t)}{T}\right) + \mathrm{i}S_{2r}\left(\frac{\pi(x-t)}{T}\right) & \text{if } m=2r+1, \quad r=1,2,\dots, \\ S_{2r}\left(\frac{\pi(x-t)}{T}\right) & \text{if } m=2r, \quad r=1,2,\dots. \end{cases} \tag{7.6}$$

[5] If $v_m(x,t)$ vanishes at some point in $(0,T)$, then $u(x,t)$ must have a pole at that same point, which is not consistent with our demand that $u(x,t)$ be sufficiently differentiable on $\mathbb{R}$. The function $\cos\frac{\pi(x-t)}{T}/\sin^m\frac{\pi(x-t)}{T}$ for $m=1,3,5,\dots$ in (7.1) vanishes at one of the points $x=t\pm T/2$ in the interval $(0,T)$, thus cannot serve as $v_m(x,t)$ when $m=1,3,5,\dots$.

Defining as before

$$K_m(t;u(\cdot,t)) = \int_0^T S_m\left(\frac{\pi(x-t)}{T}\right)u(x,t)\,dx, \tag{7.7}$$

we can express the HFP integrals

$$\int_0^T f(x,t)\,dx = \int_0^T u(x,t)v_m(x,t)\,dx \equiv \widetilde{K}_m(t;u(\cdot,t))$$

as follows:

$$\widetilde{K}_m(t;u(\cdot,t)) = \begin{cases} K_1(t;u(\cdot,t)) + \mathrm{i}\int_0^T u(x,t)\,dx & \text{if } m=1, \\ K_{2r+1}(t;u(\cdot,t)) + \mathrm{i}K_{2r}(t;u(\cdot,t)) & \text{if } m=2r+1, \quad r=1,2,\dots, \\ K_{2r}(t;u(\cdot,t)) & \text{if } m=2r, \quad r=1,2,\dots. \end{cases} \tag{7.8}$$

Noting that $\int_0^T u(x,t)\,dx$ (for $m=1$) is already a regular integral that can be computed with spectral accuracy by the trapezoidal rule, we see that we have to deal only with the individual $K_m(t;u(\cdot,t))$, $m=1,2,\dots$, which we already know how to handle via regularization and trigonometric interpolation. Denoting by $\widetilde{Q}_{m,n}(t;u(\cdot,t))$ the quadrature formulas for $\widetilde{K}_m(t;u(\cdot,t))$, we have

$$\widetilde{Q}_{m,n}(t;u(\cdot,t)) = \begin{cases} Q_{1,n}(t;u(\cdot,t)) + \mathrm{i}\frac{T}{2n}\sum_{k=0}^{2n-1} u(x_{n,k},t) & \text{if } m=1, \\ Q_{2r+1,n}(t;u(\cdot,t)) + \mathrm{i}Q_{2r,n}(t;u(\cdot,t)) & \text{if } m=2r+1, \quad r=1,2,\dots, \\ Q_{2r,n}(t;u(\cdot,t)) & \text{if } m=2r, \quad r=1,2,\dots. \end{cases} \tag{7.9}$$

Actually, by (5.8), we can express $\widetilde{Q}_{m,n}(t;u(\cdot,t))$ as

$$\widetilde{Q}_{m,n}(t;u(\cdot,t)) = \frac{1}{2n}\sum_{k=0}^{2n-1}\left[\sum_{q=-n}^{n}{}'' \widetilde{L}_{m,q}e_q(t-x_{n,k})\right]u(x_{n,k},t), \tag{7.10}$$

where

$$\widetilde{L}_{1,0} = L_{1,0} + \mathrm{i}T; \quad \widetilde{L}_{1,q} = L_{1,q} \quad \forall q \neq 0, \tag{7.11}$$

$$\widetilde{L}_{2r,q} = L_{2r,q} \quad \widetilde{L}_{2r+1,q} = L_{2r+1,q} + \mathrm{i}L_{2r,q}, \quad \forall q, \quad r=1,2,\dots. \tag{7.12}$$

Observe that we only need the $2n$ function values $u(x_{n,k},t)$, $k=0,1,\dots,2n-1$, to compute $\widetilde{Q}_{m,n}(t;u(\cdot,t))$ for every m.

Below, we will continue to use the new notation $\widetilde{K}_m(t;\cdot)$, $\widetilde{L}_{m,q}$, and $\widetilde{Q}_{m,n}(t;\cdot)$.

7.2. Application to singular integral equations

We now consider the application of the quadrature formulas $\widetilde{Q}_{m,n}(t;\cdot)$ to the numerical solution of singular integral equations of the form

$$\lambda w(t) + \int_0^T G(x,t)w(x)\,dx = \theta(t), \quad t \in [0,T], \tag{7.13}$$

which arise in boundary integral formulations of some two-dimensional boundary value problems, at least for $m=1,2$. Here the constant λ and the functions $G(x,t)$ and $\theta(x)$ are known; $w(x)$ is the unknown function, hence is the required solution to this equation. $G(x,t)$ is usually some sort of Green's function.

1. $G(x,t)$ is T-periodic both in x and in t. It is sufficiently differentiable as a function of x on $\mathbb{R} \setminus \{t \pm kT\}_{k=0}^{\infty}$ and, for $0 \le x \le T$, it is of the form

$$G(x,t) = \frac{H(x,t)}{(x-t)^m}, \quad 0 < t < T, \tag{7.14}$$

   $H(x,t)$ being sufficiently differentiable and known/computable for x and t in $[0,T]$. [Note that $H(x,t)$ is *not* T-periodic since $(x-t)^{-m}$ is not.]
2. $w(x)$ and $\theta(x)$ are T-periodic and sufficiently differentiable on $\mathbb{R}$.

Additional conditions may have to be imposed on $G(x,t)$ and/or $\theta(x)$ to ensure uniqueness of solution; we will skip this issue below.

With the functions $v_m(x,t)$ defined as in (7.3), we define $N(x,t)$ and $u(x,t)$ as

$$N(x,t) = \frac{G(x,t)}{v_m(x,t)}, \quad u(x,t) = N(x,t)w(x) \quad \Rightarrow \quad G(x,t)w(x) = u(x,t)v_m(x,t), \tag{7.15}$$

and recall that, when $H(x,t)$ is known, with $x,t \in [0,T]$, $N(x,t)$ can be computed from [cf. (7.5)]

$$N(x,t) = \left(\frac{\pi}{T}\right)^m \times \begin{cases} H(x,t)\left[\operatorname{sinc}\left(\frac{\pi(x-t)}{T}\right)\right]^m e^{-\mathrm{i}\pi(x-t)/T} & \text{if } m = 1,3,5,\dots, \\ H(x,t)\left[\operatorname{sinc}\left(\frac{\pi(x-t)}{T}\right)\right]^m & \text{if } m = 2,4,6,\dots, \end{cases} \tag{7.16}$$

with

$$N(t,t) = \left(\frac{\pi}{T}\right)^m H(t,t). \tag{7.17}$$

Let us also observe that $N(x,t)$ and $w(x,t)$ are T-periodic in x and are sufficiently differentiable, therefore, so is their product $u(x,t)$. Finally, let us rewrite (7.13) in the form

$$\lambda w(t) + \widetilde{K}_m(t; N(\cdot,t)w(\cdot)) = \theta(t). \tag{7.18}$$

We now turn to the numerical solution of this integral equation. First, we set $x_{n,k} = kT/(2n)$, $k = 0,1,\dots,2n-1$, and replace $\widetilde{K}_m(t; N(\cdot,t)w(\cdot))$ by $\widetilde{Q}_{m,n}(t; N(\cdot,t)w(\cdot))$. Next, we replace $u(x,t) = N(x,t)w(x)$ by $\phi_n(x,t)$, its trigonometric interpolant at the $2n$ points $x_{n,k}$, and replace $w(x_{n,k})$ everywhere by the approximation $\widetilde{w}_{n,k}$. Finally, we set $t = x_{n,j}$, $j = 0,1,\dots,2n-1$, everywhere. This results in the following $2n$ equations in the $2n$ unknowns $\widetilde{w}_{n,k}$:

$$\lambda \widetilde{w}_{n,j} + \frac{1}{2n}\sum_{k=0}^{2n-1}\left[\sum_{q=-n}^{n}{}'' \widetilde{L}_{m,q} e_q(x_{n,j} - x_{n,k})\right] N(x_{n,k}, x_{n,j})\widetilde{w}_{n,k} = \theta(x_{n,j}), \quad 0 \le j \le 2n-1. \tag{7.19}$$

Since the underlying numerical quadrature formula $\widetilde{Q}_{m,n}(t; N(\cdot,t)w(\cdot))$ has high accuracy, we expect the $\widetilde{w}_{n,k}$ to approximate the $w(x_{n,k})$ with high accuracy too.

8. Treatment of $\int_0^T f(x,t)\,dx$ with $m = 0$

Recall that we have also dealt with the regular integrals $K_0(t; u) = \int_0^T f(x,t)\,dx$, where $f(x,t)$ is T-periodic in x and is expressed as [recall (1.2)]

$$f(x,t) = \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right)u(x), \quad 0 < t < T, \tag{8.1}$$

$u(x)$ being T-periodic and sufficiently differentiable on $\mathbb{R}$. We now consider the regular integrals $I_0(t; f) = \int_0^T f(x,t)\,dx$ where $f(x,t)$ is T-periodic in $x \in \mathbb{R}$ and has the general form

$$f(x,t) = \Big(\log\big|\psi(x) - \psi(t)\big|\Big)w(x), \quad 0 < t < T, \tag{8.2}$$

such that $\psi(x)$ and $w(x)$ are T-periodic and sufficiently differentiable on $\mathbb{R}$, and $\psi'(x) \neq 0$ on $[0,T]$. In addition, t is being held fixed throughout. Integrals like this appear in boundary integral equation formulation of certain two-dimensional boundary value problems dealt with in some of the papers mentioned in the Introduction. Dividing and multiplying $\big|\psi(x) - \psi(t)\big|$ by $\big|\sin\frac{\pi(x-t)}{T}\big|$, we now express $f(x,t)$ in the form

$$f(x,t) = \left[H(x,t) + \left(\log\left|\sin\frac{\pi(x-t)}{T}\right|\right)\right]w(x), \tag{8.3}$$

where

$$H(x,t) = \log\left(\frac{T}{\pi}\frac{|\psi[x,t]|}{\operatorname{sinc}\left(\frac{\pi(x-t)}{T}\right)}\right), \quad \psi[x,t] = \begin{cases} \dfrac{\psi(x) - \psi(t)}{x - t} & \text{if } x \neq t, \\ \psi'(t) & \text{if } x = t. \end{cases} \tag{8.4}$$

Note that $H(x,t)$ is T-periodic and sufficiently differentiable on $\mathbb{R}$ because $\psi[x,t]$ is nonzero and of one sign on $[0,T]$. Therefore,

$$\int_0^T f(x,t)\,dx = \int_0^T H(x,t)w(x)\,dx + K_0(t;w). \tag{8.5}$$

We already know how to handle $K_0(t;w)$; we simply approximate it by $K_0(t;\phi_n)$, where $\phi_n(x)$ is the balanced trigonometric polynomial that interpolates $w(x)$ precisely as described in Section 5. The integral $\int_0^T H(x,t)w(x)\,dx$ can be evaluated by the trapezoidal rule using the interpolation points for $w(x)$ [hence for $\phi_n(x)$] as the abscissas. Thus, the combined computational cost is again $2n$ evaluations of $w(x)$ at these interpolation points. If $\psi(x)$ and $w(x)$ are infinitely differentiable on $\mathbb{R}$, then the convergence of the quadrature method is spectral.

We leave the discussion concerning the application of this approach to the solution of the relevant integral equations to the reader. Of course, this can be achieved as described in Section 7, with proper modifications.

Appendix A. Trigonometric interpolation

A.1. Construction of the trigonometric interpolant $\phi_n(x)$

Let $u(x)$ be continuous on the interval $[0,T]$. Let us denote by $\phi_n(x)$ the balanced trigonometric polynomial of degree n that interpolates $u(x)$ on $[0,T]$ at the $2n$ equidistant points $x_{n,0}, x_{n,1}, \dots, x_{n,2n-1}$ in $[0,T]$, where

$$x_{n,k} = \frac{kT}{2n}, \quad k = 0,1,\dots,2n-1. \tag{A.1}$$

Thus, with $e_q(x) = e^{\mathrm{i}2q\pi x/T}$ as before,

$$\phi_n(x) = {\sum_{q=-n}^{n}}'' \tilde{c}_{n,q} e_q(x), \quad \tilde{c}_{n,n} = \tilde{c}_{n,-n}; \quad \phi_n(x_{n,k}) = u(x_{n,k}), \quad k = 0,1,\dots,2n-1. \tag{A.2}$$

Here the double prime on the summation ${\sum''}_{q=-n}^{n}$ means that the terms with $q = \pm n$ are to be multiplied by $1/2$.[6] With these $x_{n,k}$, let us define the discrete inner product $(\cdot\,,\cdot)$ as in

$$(G\,,H) = \sum_{k=0}^{2n-1} \overline{G(x_{n,k})}\, H(x_{n,k}). \tag{A.3}$$

Then the functions $e_q(x)$ have the discrete orthogonality property

$$(e_p\,,e_q) = \sum_{k=0}^{2n-1} e^{\mathrm{i}(q-p)k\pi/n} = \begin{cases} 2n & \text{if } q-p = 2n\tau,\ \tau = 0,\pm1,\pm2,\dots, \\ 0 & \text{otherwise.} \end{cases} \tag{A.4}$$

Therefore,

$$\tilde{c}_{n,q} = \frac{(e_q\,,u)}{2n} = \frac{1}{2n}\sum_{k=0}^{2n-1} e^{-\mathrm{i}qk\pi/n} u(x_{n,k}), \quad -n \le q \le n. \tag{A.5}$$

Note that, for $\tilde{c}_{n,0}$, $\tilde{c}_{n,n}$, and $\tilde{c}_{n,-n}$, (A.5) gives

$$\tilde{c}_{n,0} = \frac{1}{2n}\sum_{k=0}^{2n-1} u(x_{n,k}); \quad \tilde{c}_{n,n} = \tilde{c}_{n,-n} = \frac{1}{2n}\sum_{k=0}^{2n-1} (-1)^k u(x_{n,k}). \tag{A.6}$$

Substituting (A.5) into (A.2), $\phi_n(x)$ can also be expressed as a trigonometric polynomial as in

[6] Note that $\tilde{c}_{n,n} = \tilde{c}_{n,-n}$ amounts to $\phi_n(x)$ being necessarily of the form

$$\phi_n(x) = \tilde{a}_0 + \sum_{q=1}^{n-1} [\tilde{a}_q \cos(qx/\mu) + \tilde{b}_q \sin(qx/\mu)] + \tilde{a}_n \cos(nx/\mu), \quad \mu = \frac{T}{2\pi},$$

$$\tilde{a}_0 = \tilde{c}_{n,0}; \quad \tilde{a}_q = \tilde{c}_{n,q} + \tilde{c}_{n,-q}, \quad \tilde{b}_q = \mathrm{i}(\tilde{c}_{n,q} - \tilde{c}_{n,-q}), \quad q = 1,\dots,n-1; \quad \tilde{a}_n = \tilde{c}_{n,\pm n}.$$

The way we have defined things here, the trigonometric interpolant $\phi_n(x)$ is said to be *balanced*. See Henrici [10, p. 43].

$$\phi_n(x) = \frac{1}{2n}\sum_{k=0}^{2n-1}\left[{\sum_{q=-n}^{n}}'' e_q(x-x_{n,k})\right]u(x_{n,k})$$
$$= \sum_{k=0}^{2n-1} D_n\big((x-x_{n,k})/\mu\big)u(x_{n,k}), \quad \mu = \frac{T}{2\pi}, \tag{A.7}$$

where $D_n(y)$ is the Dirichlet kernel given as

$$D_n(y) = \frac{1}{2n}{\sum_{q=-n}^{n}}'' e^{\mathrm{i}qy} = \frac{1}{2n}\sin(ny)\cot(y/2). \tag{A.8}$$

Note. $\tilde{c}_{n,q}$ is the trapezoidal rule approximation to the integral representation of the Fourier coefficient c_q of $u(x)$, namely, $c_q = \frac{1}{T}\int_0^T \overline{e_q(x)}u(x)\,dx$. Therefore, $\tilde{c}_{n,0}$ is the trapezoidal rule approximation to the integral $c_0 = \frac{1}{T}\int_0^T u(x)\,dx$.

The following lemma, whose proof can be achieved by invoking (A.4), concerns the reproducing property of trigonometric interpolation as defined above.

Lemma A.1. *As defined above via* (A.1)–(A.5)*, trigonometric interpolation reproduces balanced trigonometric polynomials of degree at most n in the following sense: if* $u(x) = {\sum}''^{\,n}_{q=-n} c_q e_q(x)$ *with* $c_n = c_{-n}$*, then* $\phi_n(x) \equiv u(x)$*.*

Throughout this appendix, we will assume that the function $u(x)$ is T-periodic and in the Hölder class $C^{M+1,\alpha}(0,T)$, $0<\alpha\leq 1$. (See footnote 3.) As a result, $u^{(s)}(x)$, $s = 0,1,\ldots,M$, have Fourier series that converge absolutely and uniformly on $\mathbb{R}$. (See subsection 4.2.)

We will also be using the notation ${\sum}''_{|p|\leq n}$ and ${\sum}''_{|q|\geq n}$ to mean that the terms with $p=\pm n$ and $q = \pm n$, respectively, are to be multiplied by $1/2$.

A.2. A general summation lemma

The following well known summation lemma is useful in the convergence analysis of the trigonometric interpolation polynomial $\phi_n(x)$ and of its derivatives and of $Q_{m,n}(t;u) = K_m(t;\phi_n)$. In this lemma, the c_q are arbitrary constants. We provide an independent proof of it here.

Lemma A.2. *Assume that* $\sum_{q=-\infty}^{\infty} c_q$ *converges absolutely. Then, for arbitrary n,*

$${\sum_{|p|\leq n}}'' \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}| = {\sum_{|q|\geq n}}'' |c_q|. \tag{A.9}$$

That is, the left-hand side of (A.9) *is a rearrangement of the right-hand side.*

Proof. We start by rewriting the double summation in the form

$${\sum_{|p|\leq n}}'' \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}| = \frac{1}{2}\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty}\big(|c_{n+2n\tau}| + |c_{-n+2n\tau}|\big) + \sum_{|p|\leq n-1}\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty}|c_{p+2n\tau}|, \tag{A.10}$$

separating the contributions of the terms with $p = \pm n$ from those with $|p|\leq n-1$.

We first note that

$$\frac{1}{2}\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty}\big(|c_{n+2n\tau}| + |c_{-n+2n\tau}|\big) = \frac{1}{2}\big(|c_{-n}| + |c_n|\big) + \sum_{j=1}^{\infty}\big(|c_{(2j+1)n}| + |c_{-(2j+1)n}|\big)$$

is the sum of all those terms in ${\sum}''_{|q|\geq n}|c_q|$ with $q = \pm(2j+1)n$, $j = 0,1,2,\ldots$.

Next, we claim that the double sum on the right-hand side of (A.10) is actually the sum of the remaining terms in ${\sum}''_{|q|\geq n}|c_q|$. This follows from the fact that for every integer q, $q \neq \pm(2j+1)n$, $j = 0,1,\ldots$, there exist unique integers τ and p, $\tau \in \{\pm 1, \pm 2, \ldots\}$ and $-(n-1)\leq p \leq n-1$, such that $q = p + 2n\tau$.

This completes the proof of (A.9). ■

A.3. Convergence of $\phi_n(x)$ and its derivatives

Consider the Fourier series of $u(x)$, namely,

$$u(x) \sim \sum_{q=-\infty}^{\infty} c_q e_q(x), \quad c_q = \frac{1}{T}\int_0^T \overline{e_q(x)}u(x)\,dx. \tag{A.11}$$

Then, by (A.5), there holds

$$\tilde{c}_{n,p} = \frac{(e_p\,, u)}{2n} = \frac{1}{2n}\left(e_p, \sum_{q=-\infty}^{\infty} c_q e_q\right) = \sum_{q=-\infty}^{\infty} c_q \frac{(e_p\,, e_q)}{2n},$$

which, upon invoking (A.4), gives

$$\tilde{c}_{n,p} = \sum_{\tau=-\infty}^{\infty} c_{p+2n\tau} = c_p + \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}, \quad -n \le p \le n. \tag{A.12}$$

Theorem A.3 that follows concerns the convergence of the sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ to $u(x)$ and is well known. (See, for example, Kress [11], [12]. See also Gaier [6], Henrici [9], [10], and Gander, Gander, and Kwok [7].)

Theorem A.3. *The sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ converges to $u(x)$ absolutely and uniformly on $[0, T]$. Actually, there holds*

$$\max_{0\le x\le 2\pi} |\phi_n(x) - u(x)| \le 2 \sum_{|q|\ge n}{}'' |c_q|. \tag{A.13}$$

Proof. We start by observing that

$$\begin{aligned}
\phi_n(x) - u(x) &= \sum_{|p|\le n}{}'' \tilde{c}_{n,p}\, e_p(x) - \sum_{q=-\infty}^{\infty} c_q e_q(x)\\
&= \sum_{|p|\le n}{}'' \left(c_p + \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\right) e_p(x) - \sum_{q=-\infty}^{\infty} c_q e_q(x)\\
&= \sum_{|p|\le n}{}'' \left(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\right) e_p(x) - \sum_{|q|\ge n}{}'' c_q e_q(x),
\end{aligned} \tag{A.14}$$

which, upon taking moduli, gives

$$\max_{0\le x\le 2\pi} |\phi_n(x) - u(x)| \le \sum_{|p|\le n}{}'' \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}| + \sum_{|q|\ge n}{}'' |c_q|. \tag{A.15}$$

The result in (A.13) follows by invoking Lemma A.2. ■

Theorem A.4 that follows concerns the convergence of the sequence $\{\phi_n^{(s)}(x)\}_{n=1}^{\infty}$ to $u^{(s)}(x)$, $s = 1, 2, \ldots$. Recall that for $s \ge 1$, $\phi_n^{(s)}(x)$ does *not* interpolate $u^{(s)}(x)$ at the points $x_{n,k}$ in (A.1).

Theorem A.4. *For $s \in \{1, 2, \ldots, M\}$, the sequence $\{\phi_n^{(s)}(x)\}_{n=1}^{\infty}$ converges to $u^{(s)}(x)$ absolutely and uniformly on $[0, T]$. Actually, there holds*

$$\max_{x\in[0,T]} |\phi_n^{(s)}(x) - u^{(s)}(x)| \le \sum_{|q|\ge n}{}'' |c_q^{(s)}|(1 + |n/q|^s), \tag{A.16}$$

where $c_q^{(s)} = c_q(\mathrm{i}q/\mu)^s$ are the Fourier coefficients of $u^{(s)}(x)$, $s = 1, 2, \ldots$.

Proof. Differentiating $\phi_n(x) = \sum''_{|p|\le n} \tilde{c}_{n,p}\, e_p(x)$ s times and invoking $u^{(s)}(x) = \sum_{q=-\infty}^{\infty} c_q^{(s)} e_q(x)$, we obtain

$$\begin{aligned}\phi_n^{(s)}(x) - u^{(s)}(x) &= \sum_{|p|\le n}'' \tilde{c}_{n,p}(\mathrm{i}p/\mu)^s e_p(x) - \sum_{q=-\infty}^{\infty} c_q(\mathrm{i}q/\mu)^s e_q(x) \\ &= \sum_{|p|\le n}'' \Bigg(c_p + \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\Bigg)(\mathrm{i}p/\mu)^s e_p(x) - \sum_{q=-\infty}^{\infty} c_q(\mathrm{i}q/\mu)^s e_q(x) \\ &= \sum_{|p|\le n}'' \Bigg(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\Bigg)(\mathrm{i}p/\mu)^s e_p(x) - \sum_{|q|\ge n}'' c_q(\mathrm{i}q/\mu)^s e_q(x),\end{aligned}$$

which, upon taking moduli, gives

$$\max_{x\in[0,T]} |\phi_n^{(s)}(x) - u^{(s)}(x)| \le \sum_{|p|\le n}'' \Bigg(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Bigg)|p/\mu|^s + \sum_{|q|\ge n}'' |c_q||q/\mu|^s. \tag{A.17}$$

Next,

$$\begin{aligned}\sum_{|p|\le n}'' \Bigg(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Bigg)|p/\mu|^s &\le (n/\mu)^s\Bigg(\sum_{|p|\le n}'' \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Bigg) \\ &= (n/\mu)^s \sum_{|q|\ge n}'' |c_q| \quad \text{by Lemma A.2} \\ &= \sum_{|q|\ge n}'' |c_q||q/\mu|^s\, |n/q|^s.\end{aligned} \tag{A.18}$$

Substituting (A.18) into (A.17), we obtain

$$\max_{x\in[0,T]} |\phi_n^{(s)}(x) - u^{(s)}(x)| \le \sum_{|q|\ge n}'' |c_q||q/\mu|^s(1 + |n/q|^s)$$

This completes the proof. ■

Combining the results of these two theorems, we reach the following conclusions:

1. In case M is finite, we have

$$\max_{x\in[0,T]} |\phi_n^{(s)}(x) - u^{(s)}(x)| = O(n^{-M-\alpha+s}) \quad \text{as } n\to\infty, \quad s = 0, 1, \dots, M. \tag{A.19}$$

2. In case $M = \infty$, we have

$$\max_{x\in[0,T]} |\phi_n^{(s)}(x) - u^{(s)}(x)| = o(n^{-\lambda}) \quad \text{as } n\to\infty \quad \forall\lambda > 0, \quad s = 0, 1, \dots, \tag{A.20}$$

   that is, we have spectral convergence.

3. In case $u(z)$ is T-periodic and analytic in an infinite strip D_ρ of the complex z-plane, with

$$D_\rho = \{z \in \mathbb{C} : |\boldsymbol{Im}\, z| < \rho\}, \tag{A.21}$$

   we have

$$|c_q^{(s)}| = O(e^{-(\theta/\mu)|q|}) \quad \text{as } q\to\pm\infty \quad \forall\theta\in(0,\rho), \tag{A.22}$$

   hence the result in (A.20) improves to read

$$\max_{x\in[0,T]} |\phi_n^{(s)}(x) - u^{(s)}(x)| = O(e^{-(\theta/\mu)n}) \quad \text{as } n\to\infty \quad \forall\theta\in(0,\rho), \quad s = 0, 1, \dots. \tag{A.23}$$

References

[1] K.E. Atkinson, A discrete Galerkin method for first kind integral equations with a logarithmic kernel, J. Integral Equ. Appl. 1 (1988) 343–363.
[2] K.E. Atkinson, An Introduction to Numerical Analysis, second edition, John Wiley & Sons Inc., New York, 1989.
[3] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge Monographs on Applied and Computational Mathematics, vol. 4, Cambridge University Press, Cambridge, 1997.
[4] P.J. Davis, P. Rabinowitz, Methods of Numerical Integration, second edition, Academic Press, New York, 1984.
[5] G. Evans, Practical Numerical Integration, Wiley, New York, 1993.
[6] D. Gaier, Ableitungsfreie Abschätzungen bei trigonometrischer Interpolation und Konjugierten-Bestimmung, Computing 12 (1974) 145–148.
[7] W. Gander, M.J. Gander, F. Kwok, Scientific Computing: An Introduction Using Maple and MATLAB, Texts in Computational Science and Engineering, vol. 11, Springer, New York, 2014.
[8] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1980, Forth printing 1983.
[9] P. Henrici, Pointwise error estimates for trigonometric interpolation and conjugation, J. Comput. Appl. Math. 8 (1982) 131–132.
[10] P. Henrici, Applied and Computational Complex Analysis, vol. 3, Wiley, New York, 1986.
[11] R. Kress, Ein ableitungsfreies Restglied für die trigonometrische Interpolation periodischer analytischer Funktionen, Numer. Math. 16 (1971) 389–396.
[12] R. Kress, Über die numerische Berechnung konjugierter Funktionen, Computing 10 (1972) 177–187.
[13] R. Kress, On the numerical solution of a hypersingular integral equation in scattering theory, J. Comput. Appl. Math. 61 (1995) 345–360.
[14] R. Kress, I.H. Sloan, On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation, Numer. Math. 66 (1993) 199–214.
[15] A.R. Krommer, C.W. Ueberhuber, Computational Integration, SIAM, Philadelphia, 1998.
[16] R. Kussmaul, Ein numerisches Verfahren zur Lösung des Neumannschen Aussenraumproblems für die Helmholtzsche Schwingungsgleichung, Computing 4 (1969) 246–273.
[17] P.K. Kythe, M.R. Schäferkotter, Handbook of Computational Methods for Integration, Chapman & Hall/CRC Press, New York, 2005.
[18] W. McLean, Fully-discrete collocation methods for an integral equation of the first kind, J. Integral Equ. Appl. 6 (1994) 537–571.
[19] W. McLean, S. Prössdorf, W.L. Wendland, A fully discrete trigonometric collocation method, J. Integral Equ. Appl. 5 (1993) 103–129.
[20] G. Monegato, Definitions, properties and applications of finite-part integrals, J. Comput. Appl. Math. 229 (2009) 425–439.
[21] J. Saranen, I.H. Sloan, Quadrature methods for logarithmic-kernel integral equations on closed curves, IMA J. Numer. Anal. 12 (1992) 167–187.
[22] A. Sidi, Euler–Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities, Math. Comput. 81 (2012) 2159–2173.
[23] A. Sidi, Compact numerical quadrature formulas for hypersingular integrals and integral equations, J. Sci. Comput. 54 (2013) 145–176.
[24] A. Sidi, Unified compact numerical quadrature formulas for Hadamard finite parts of singular integrals of periodic functions, Calcolo 58 (2021), Article number 22.
[25] A. Sidi, Exactness and convergence properties of some recent numerical quadrature formulas for supersingular integrals of periodic functions, Calcolo 58 (2021), Article number 36.
[26] A. Sidi, PVTSI$^{(m)}$: a novel approach to computation of Hadamard finite parts of nonperiodic singular integrals, Calcolo 59 (2022), Article number 7.
[27] A. Sidi, Exponential convergence of some recent numerical quadrature methods for Hadamard finite parts of singular integrals of periodic analytic functions, Calcolo (2022), in press.
[28] I.H. Sloan, B.J. Burn, An unconventional quadrature method for logarithmic-kernel integral equations on closed curves, J. Integral Equ. Appl. 4 (1992) 117–151.
[29] Y. Yan, I.H. Sloan, On integral equations of the first kind with logarithmic kernels, J. Integral Equ. Appl. 1 (1988) 549–579.
[30] A. Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, 1959.