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# Spectrally accurate numerical quadrature formulas for a class of algebraically singular periodic Hadamard finite part integrals by regularization

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ABSTRACT

We consider the numerical computation of Hadamard Finite Part (HFP) integrals

$$H_\sigma(t;u)=\,=\!\!\!\!\!\!\int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx,\quad 0<t<T,\quad \sigma<-1,\quad \sigma\notin\mathbb{Z},$$

$u(x)$ being sufficiently differentiable and $T$-periodic on $\mathbb{R}$. Thus $\sigma=-(m+\delta)$, $m\in\{1,2,3,\ldots\}$, $0<\delta<1$. For each such $\sigma$, we regularize $H_\sigma(t;u)$, and show that

$$H_\sigma(t;u)=H_{\sigma+2r}(t;U_\sigma),\quad r=\lfloor(m+1)/2\rfloor,$$

where $U_\sigma(x)=\sum_{k=0}^{r}a_k u^{(2k)}(x)$ for some constants $a_k$, $H_{\sigma+2r}(t;U_\sigma)$ being a regular integral. We then propose to approximate $H_\sigma(t;u)$ by the quadrature formula $Q_{\sigma,n}(t;u)\equiv H_\sigma(t;\phi_n)$, where $\phi_n(x)$ is the $n^{\text{th}}$-order balanced trigonometric polynomial that interpolates $u(x)$ on $[0,T]$ at the $2n$ equidistant points $x_{n,k}=\frac{kT}{2n}$, $k=0,1,\ldots,2n-1$. The implementation of $Q_{\sigma,n}(t;u)$ is simple, the only input needed for this being the $2n$ function values $u(x_{n,k})$, $k=0,1,\ldots,2n-1$. Using Fourier analysis techniques, we develop a complete convergence theory for $Q_{\sigma,n}(t;u)$ as $n\to\infty$ and prove that it enjoys spectral convergence when $u\in C^\infty(\mathbb{R})$.

We also show that the theoretical developments and numerical quadrature formulas developed for the HFP integrals $H_\sigma(t;u)$ with $\sigma<-1$ and $\sigma\notin\mathbb{Z}$ apply, with no changes, to the regular singular integrals $H_\sigma(t;u)$ with $\sigma>-1$ and $\sigma\notin\mathbb{Z}$.

We illustrate the effectiveness of $Q_{\sigma,n}(t;u)$ with numerical examples both for $\sigma<-1$ and $\sigma>-1$.

Finally, we show that the HFP or regular integral $=\!\!\!\!\!\!\int_0^T f(x)\,dx$ of any $T$-periodic integrand $f(x)$ that has algebraic singularities of the form $|x-t+kT|^\sigma$, $0<t<T$, $k=0,\pm1,\pm2,\ldots$, with $\sigma\notin\mathbb{Z}$, but is sufficiently differentiable in $x$ on $\mathbb{R}\setminus\{t\pm kT\}_{k=0}^\infty$, can be expressed as $H_\sigma(t;u)$, where $u(x)$ is a $T$-periodic and sufficiently differentiable function of $x$ on $\mathbb{R}$ that can be computed from $f(x)$. Therefore, $=\!\!\!\!\!\!\int_0^T f(x)\,dx$ can be computed efficiently using our new numerical quadrature formulas $Q_{\sigma,n}(t;u)$ on the individual $H_\sigma(t;u)$. Again, only $2n$ function evaluations, namely, $u(x_{n,k})$, $k=0,1,\ldots,2n-1$, are needed for the whole process.



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## 1. Introduction and background

In this work, we consider the efficient numerical computation of $T$-periodic *Hadamard Finite Part (HFP)* integrals of the form

$$H_\sigma(t;u) = \,=\!\!\!\!\!\!\int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx, \quad 0<t<T, \quad \sigma<-1, \quad \sigma\notin\mathbb{Z}, \tag{1.1}$$

[1] $u(x)$ being sufficiently differentiable and $T$-periodic on $\mathbb{R}$, noting also that the factor $\left|\sin\frac{\pi(x-t)}{T}\right|^\sigma$ is $T$-periodic as well. Clearly, when $\sigma < -1$, these integrals are not defined in the regular sense since their integrands have *algebraic* singularities of the form $|x-t|^\sigma$ in $(0,T)$, which are *not* integrable. They *are* defined in the sense of Hadamard Finite Part (HFP), however.

In this work, we study the analytical properties of $H_\sigma(t;u)$ and derive simple and efficient numerical quadrature formulas for them. We approach the analysis and approximation of $H_\sigma(t;u)$ in Sections 2–4, in two major steps:

- Expressing $\sigma$ in the form $\sigma = -(m+\delta)$, $m\in\{1,2,3,\ldots\}$ and $0<\delta<1$, we begin by regularizing the *divergent* integral

  $$\int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx$$

  in the sense that $H_\sigma(t;u) = H_\eta(t;U_\sigma)$, where $U_\sigma(x)$ is some linear combination of $u^{(2i)}(x)$, $i=0,1,\ldots,r$, where $r=\lfloor (m+1)/2\rfloor$, and

  $$\eta = \begin{cases} -\delta & \text{when } m=2r \\ 1-\delta & \text{when } m=2r-1 \end{cases} \quad \Rightarrow \quad \eta>-1,$$

  hence $H_\eta(t;U_\sigma)$ is actually a *regular* integral, that is,

  $$H_\sigma(t;u) = H_\eta(t;U_\sigma) = \int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\eta U_\sigma(x)\,dx.$$

  This is done in Section 2. In Section 3, we construct the Fourier series for $H_\sigma(t;u)$ by making use of the developments of Section 2.
- In Section 4, we develop our numerical quadrature formula for $H_\sigma(t;u)$ as follows: We interpolate $u(x)$ at $2n$ equidistant points in $[0,T]$ by a balanced trigonometric polynomial $\phi_n(x)$ and take $H_\sigma(t;\phi_n)$ as our approximation to $H_\sigma(t;u)$. Thus, the cost of computing $H_\sigma(t;\phi_n)$ is only $2n$ evaluations of $u(x)$, no derivative information being needed. For $u\in C^P(\mathbb{R})$, $P>m$, we show by using Fourier analysis techniques that $\lim_{n\to\infty} H_\sigma(t;\phi_n) = H_\sigma(t;u)$. We also provide the rate of convergence and prove that the accuracy of $H_\sigma(t;\phi_n)$ as an approximation to $H_\sigma(t;u)$ increases as $P$ increases. This accuracy is spectral when $P=\infty$, thus when $u\in C^\infty(\mathbb{R})$.

In Section 5, we show that the developments of Sections 2–4 can be extended with no difficulty to the *regular* singular integrals

$$H_\sigma(t;u) = \int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx, \quad 0<t<T, \quad \sigma>-1, \quad \sigma\notin\mathbb{Z}. \tag{1.2}$$

In Section 6, we present several numerical examples that demonstrate the efficiency of our numerical quadrature formulas, both for divergent HFP integrals (with $\sigma<-1, \sigma\notin\mathbb{Z}$) and for regular singular integrals (with $\sigma>-1, \sigma\notin\mathbb{Z}$). In Section 7, we discuss their application to periodic singular integrals and integral equations whose integrands have an *algebraic* singularity of the form $|x-t|^\sigma$, $\sigma\notin\mathbb{Z}$, but are not necessarily given as in (1.1) or (1.2).

We have verified the validity of our approach by treating the examples of Section 6 numerically via a completely different approach that uses a generalized Euler–Maclaurin expansion and Richardson extrapolation. This is explained in detail in the appendix to this work.

We note that integrals involving $H_\sigma(t;\cdot)$ arise in a natural way, for example, when dealing with Cauchy-like transforms of a function $w(\zeta)$ on the unit circle, namely,

$$J_\sigma(z;w) = \,=\!\!\!\!\!\!\int_\Gamma |\zeta-z|^\sigma w(\zeta)\,d\zeta, \quad z\in\Gamma=\{\zeta:|\zeta|=1\}, \quad \sigma<-1, \quad \sigma\notin\mathbb{Z},$$

$\Gamma$ being positively oriented. Making the substitution $\zeta=e^{\mathrm{i}x}$, $0\le x\le 2\pi$, so that $T=2\pi$, and noting that $z=e^{\mathrm{i}t}$ for some unique $t\in[0,2\pi)$, and denoting $\widehat{w}(x) = \mathrm{i}2^\sigma e^{\mathrm{i}x} w(e^{\mathrm{i}x})$, $J_\sigma(z;w)$ can be expressed as

$$J_\sigma(z;w) = \,=\!\!\!\!\!\!\int_0^{2\pi} \left|\sin\frac{x-t}{2}\right|^\sigma \widehat{w}(x)\,dx = H_\sigma(t;\widehat{w}).$$

[1] In the sequel, we will use the standard notation $\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\ldots\}$.

Before proceeding to the next section, we would like to mention that, in a recent paper [1], we studied and developed numerical quadrature formulas for $T$-periodic HFP integrals of the form

$$K_m(t;u) = \oint_0^T \frac{\cos\frac{\pi(x-t)}{T}}{\sin^m\frac{\pi(x-t)}{T}}u(x)\,dx, \qquad 0<t<T, \quad \text{if } m=1,3,5,\ldots, \tag{1.3a}$$

$$K_m(t;u) = \oint_0^T \frac{1}{\sin^m\frac{\pi(x-t)}{T}}u(x)\,dx, \qquad 0<t<T, \quad \text{if } m=2,4,6,\ldots, \tag{1.3b}$$

$u(x)$ being sufficiently differentiable and $T$-periodic on $\mathbb{R}$. Clearly, these integrals are not defined in the regular sense, too, since their integrands have *polar* (as opposed to *algebraic*) singularities of the form $(x-t)^{-m}$ on $(0,T)$, which are not integrable. They *are* defined in the sense of HFP, however. As we will see later in this work, the HFP integrals $H_\sigma(t;u)$, due to the fact $\sigma$ is not an integer, are of a completely different nature than the HFP integrals $K_m(t;u)$.[2]

Our approach to the treatment of the HFP integrals $H_\sigma(t;u)$ in this work resembles somewhat the approach we took in [1] to the HFP integrals $K_m(t;u)$. Mathematically, it is quite different and more complicated, however.

For HFP integrals, we refer the reader to the books by Davis and Rabinowitz [2], Evans [3], Krommer and Ueberhuber [4], and Kythe and Schäferkotter [5]. See also the paper [6] by Monegato for a review. For trigonometric interpolation, see the books by Atkinson [7], Henrici [8], and Zygmund [9], for example. See also [1, Appendix].

Finally, we would like to draw the attention of the reader to the papers [10–13] by the author that treat the periodic HFP integrals $K_m(t;\cdot)$ via the author's generalized Euler–Maclaurin expansion, which can be found in [14].

## 2. Regularization of $H_\sigma(t;u)$

We begin by recalling that if a function $h(x)$ has a nonintegrable singularity at $x=t$ for $t\in(a,b)$ but is integrable on any subinterval of $[a,b]$ that does not contain $x=t$, then $\oint_a^b h(x)\,dx$, the HFP of $\int_a^b h(x)\,dx$, is obtained by expanding

$$\Lambda(\epsilon) = \int_a^{t-\epsilon} h(x)\,dx + \int_{t+\epsilon}^b h(x)\,dx, \quad \epsilon>0,$$

asymptotically as $\epsilon\to 0+$, the asymptotic expansion containing terms of the form $\epsilon^p(\log\epsilon)^q$ with arbitrary $p$ and integer $q$, and by discarding those terms that go to infinity and those that go to zero, and retaining the limit as $\epsilon\to 0+$ of the remaining terms. (See Monegato [6], for example.) We will make use of this approach to the HFP next. Clearly, if $\int_a^b h(x)\,dx$ exists as a regular integral, then $\lim_{\epsilon\to 0+}\Lambda(\epsilon)=\int_a^b h(x)\,dx$; therefore, $\oint_a^b h(x)\,dx = \int_a^b h(x)\,dx$ in such cases.

In the sequel, we will also make use of the short-hand notation

$$\theta = \frac{T}{\pi}. \tag{2.1}$$

**Theorem 2.1.** *Let*

$$\sigma = -(m+\delta), \quad m\in\{1,2,3,\ldots\}, \quad 0<\delta<1, \tag{2.2}$$

*and*

$$\widehat{m} = \begin{cases} m & \text{if } m \text{ even,} \\ m+1 & \text{if } m \text{ odd,}\end{cases} \quad \Rightarrow \quad \widehat{m}\geq 2 \quad \text{and even.} \tag{2.3}$$

*Then, provided $u\in C^{\widehat{m}}(\mathbb{R})$ and is $T$-periodic, there holds*

$$H_\sigma(t;\widetilde{u}_0) = H_{\sigma+2}(t;\widetilde{u}_1), \tag{2.4}$$

*where*

$$\widetilde{u}_0(x)\equiv u(x) \quad \text{and} \quad \widetilde{u}_1(x) = \frac{\sigma+2}{\sigma+1}\widetilde{u}_0(x) + \frac{\theta^2}{(\sigma+1)(\sigma+2)}\widetilde{u}_0''(x). \tag{2.5}$$

**Remark.** On comparing the two HFP integrals $H_{\sigma+2}(t;\widetilde{u}_1)$ and $H_\sigma(t;\widetilde{u}_0)$, we realize that the former is less singular than the latter. By applying Theorem 2.1 a number of times depending on $\sigma$, we end up with a regular integral. We will discuss this in more detail following the proof of this theorem.

[2] Note that when $\sigma=-2r$, $r=1,2,3,\ldots$, we have $H_{-2r}(t;u)=K_{2r}(t;u)$, hence we need not concern ourselves with these integrals as this has been done in [1]. In addition, the treatment of the integrals $K_m(t;u)$, $m=1,2,\ldots$, in [1] is necessarily very different from the treatment of the integrals $H_\sigma(t;u)$ with $\sigma\notin\mathbb{Z}$ in the present work.

**Proof.** Let us make the change of variable $y = \pi(x - t)/T$ in the integral representation of $H_\sigma(t;u)$ in (1.1). Then we have

$$\begin{aligned} H_\sigma(t;u) &= \frac{T}{\pi} \,\rlap{\,=}\int_{-\pi t/T}^{\pi - \pi t/T} |\sin y|^\sigma\, w(y)\, dy, \quad w(y) \equiv u\left(t + \frac{T}{\pi} y\right) \\ &= \frac{T}{\pi} \,\rlap{\,=}\int_{-\pi/2}^{\pi/2} |\sin y|^\sigma\, w(y)\, dy \\ &= \frac{T}{\pi} \,\rlap{\,=}\int_{0}^{\pi} (\sin y)^\sigma\, w(y)\, dy \end{aligned} \tag{2.6}$$

since (i) by Sidi [15, Theorem 3.2], $H_\sigma(t;u)$ is invariant under any legitimate linear or nonlinear variable transformation when $\sigma < -1$ and is not an integer, (ii) the integrand $|\sin y|^\sigma w(y)$ is $\pi$-periodic because both $|\sin y|^\sigma$ and $w(y)$ are $\pi$-periodic, and (iii) $\sin y \geq 0$ for $0 \leq y \leq \pi$. Of course, $w \in C^{\widehat{m}}(\mathbb{R})$ since $u \in C^{\widehat{m}}(\mathbb{R})$. For simplicity of notation, let us define

$$\widehat{H}_\sigma(w) = \rlap{\,=}\int_0^\pi (\sin y)^\sigma\, w(y)\, dy = \rlap{\,=}\int_{-\pi/2}^{\pi/2} |\sin y|^\sigma\, w(y)\, dy. \tag{2.7}$$

Then, (2.6) becomes

$$H_\sigma(t;u) = \theta \widehat{H}_\sigma(w). \tag{2.8}$$

In addition,

$$w^{(s)}(y) = \theta^s u^{(s)}(t + \theta y), \quad H_\sigma(t; u^{(s)}) = \theta^{1-s} \widehat{H}_\sigma(w^{(s)}), \quad s = 1, 2, \ldots. \tag{2.9}$$

We now begin the process of regularizing $\widehat{H}_\sigma(w)$. Normally, regularization of a singular integral is achieved by integration by parts as many times as needed.[3] Direct application of this approach is problematic in our case for the reason that we do not want to compromise the $\pi$-periodicity of any part of the integrand in the HFP integral that follows from the regularization process; this is most crucial.

To achieve this goal, we begin by expressing $\widehat{H}_\sigma(w)$ given as in (2.7) in the form

$$\widehat{H}_\sigma(w) = \rlap{\,=}\int_0^\pi (\sin y)^\sigma (\sin^2 y + \cos^2 y) w(y)\, dy,$$

noting that the new integrand is still $\pi$-periodic because both $\sin^2 y$ and $\cos^2 y$ are $\pi$-periodic. We thus have

$$\begin{aligned} \widehat{H}_\sigma(w) &= \widehat{H}_{\sigma+2}(w) + E, \\ E &= \rlap{\,=}\int_0^\pi (\sin y)^\sigma (\cos y)^2 w(y)\, dy. \end{aligned} \tag{2.10}$$

Note that $\widehat{H}_{\sigma+2}(w)$ is already less singular than $\widehat{H}_\sigma(w)$. Therefore, we need to deal only with $E$. We first note that

$$\begin{aligned} E &= \rlap{\,=}\int_0^\pi [(\sin y)^\sigma \cos y]\, [\cos y\, w(y)]\, dy \\ &= \rlap{\,=}\int_0^\pi \left[\frac{d}{dy} \frac{(\sin y)^{\sigma+1}}{\sigma + 1}\right] [\cos y\, w(y)]\, dy, \end{aligned} \tag{2.11}$$

which, upon integrating by parts, gives

$$\begin{aligned} E &= M_1 - \rlap{\,=}\int_0^\pi \frac{(\sin y)^{\sigma+1}}{\sigma+1} [-\sin y\, w(y) + \cos y\, w'(y)]\, dy, \\ M_1 &= \rlap{\,=}\int_0^\pi \frac{d}{dy}\left[\frac{(\sin y)^{\sigma+1}}{\sigma+1} \cos y\, w(y)\right] dy, \end{aligned} \tag{2.12}$$

---

[3] Suppose we want to regularize the HFP integral

$$I(t) = \rlap{\,=}\int_a^b |x - t|^\sigma g(x)\, dx, \quad a < t < b, \quad \sigma = -(m + \delta), \quad m \in \{1, 2, 3, \ldots\}, \quad 0 < \delta < 1, \quad g \in C^m[a, b].$$

Writing $I(t) = \rlap{\,=}\int_a^t (t-x)^\sigma g(x)\, dx + \rlap{\,=}\int_t^b (x-t)^\sigma g(x)\, dx$, and applying integration by parts to the regular integrals $\int_a^{t-\epsilon}$ and $\int_{t+\epsilon}^b$, and proceeding as in the first paragraph of Section 2, we obtain

$$\begin{aligned} I(t) &= \frac{(t-a)^{\sigma+1} g(a) + (b-t)^{\sigma+1} g(b)}{\sigma+1} + I_1(t), \\ I_1(t) &= \rlap{\,=}\int_a^b |x - t|^{\sigma+1} g_1(x)\, dx, \quad g_1(x) = \begin{cases} g'(x)/(\sigma+1) & \text{if } x < t, \\ -g'(x)/(\sigma+1) & \text{if } x > t. \end{cases} \end{aligned}$$

Clearly, $I_1(t)$ is less singular than $I(t)$ and $g_1 \in C^{m-1}[a, b] \setminus \{t\}$.

and, by (2.7),

$$E = M_1 + \frac{1}{\sigma+1}\widehat{H}_{\sigma+2}(w) - F, \tag{2.13}$$

where

$$\begin{aligned} F &= \fint_0^\pi \left[\frac{(\sin y)^{\sigma+1}}{\sigma+1}\cos y\right] w'(y)\,dy \\ &= \fint_0^\pi \left[\frac{d}{dy}\frac{(\sin y)^{\sigma+2}}{(\sigma+1)(\sigma+2)}\right] w'(y)\,dy. \end{aligned} \tag{2.14}$$

Applying integration by parts to the integral representing $F$, as before, we have

$$\begin{aligned} F &= M_2 - \frac{1}{(\sigma+1)(\sigma+2)}\widehat{H}_{\sigma+2}(w''), \\ M_2 &= \fint_0^\pi \frac{d}{dy}\left[\frac{(\sin y)^{\sigma+2}}{(\sigma+1)(\sigma+2)} w'(y)\right] dy. \end{aligned} \tag{2.15}$$

Combining now (2.11)–(2.15) in (2.10), we finally obtain

$$\widehat{H}_\sigma(w) = \frac{\sigma+1}{\sigma+2}\widehat{H}_{\sigma+2}(w) + \frac{1}{(\sigma+2)(\sigma+1)}\widehat{H}_{\sigma+2}(w'') + M \tag{2.16}$$

with

$$\begin{aligned} M &= M_1 - M_2 = \fint_0^\pi \frac{d}{dy}\psi(y)\,dy = \fint_0^\pi \psi'(y)\,dy, \\ \psi(y) &= \frac{(\sin y)^\sigma}{\sigma+1}\left[\frac{\sin 2y\, w(y)}{2} - \frac{\sin^2 y\, w'(y)}{\sigma+2}\right] \quad \text{when } 0 \le y \le \pi. \end{aligned} \tag{2.17}$$

We now need to show that $M = 0$.

Let us express $M$ in the form

$$M = \fint_0^{\pi/2} \psi'(y)\,dy + \fint_{\pi/2}^{\pi} \psi'(z)\,dz.$$

Making the change of variable $z = y + \pi$ in the second integral, we obtain

$$M = \fint_0^{\pi/2} \psi'(y)\,dy + \fint_{-\pi/2}^{0} \psi'(y+\pi)\,dy. \tag{2.18}$$

Observing now that, because $\sin 2y$, $\sin^2 y$, $w(y)$, and $w'(y)$ are all $\pi$-periodic, we have from (2.17) that

$$\psi(y+\pi) = \frac{(-\sin y)^\sigma}{\sigma+1}\left[\frac{\sin 2y\, w(y)}{2} - \frac{\sin^2 y\, w'(y)}{\sigma+2}\right] \quad \text{when } -\pi/2 \le y \le 0,$$

and, therefore, (2.18) becomes

$$\begin{aligned} M &= \fint_{-\pi/2}^{\pi/2} \widehat{\psi}\,'(y)\,dy, \\ \widehat{\psi}(y) &= \frac{|\sin y|^\sigma}{\sigma+1}\left[\frac{\sin 2y\, w(y)}{2} - \frac{\sin^2 y\, w'(y)}{\sigma+2}\right] \quad \text{when } -\pi/2 \le y \le \pi/2. \end{aligned} \tag{2.19}$$

Because $\widehat{\psi}\,'(y)$ has a nonintegrable singularity (only) at $y = 0$, we must analyze

$$\begin{aligned} \Lambda(\epsilon) &= \left[\fint_{-\pi/2}^{-\epsilon} + \fint_{\epsilon}^{\pi/2}\right]\widehat{\psi}\,'(y)\,dy \\ &= [\widehat{\psi}(\pi/2) - \widehat{\psi}(-\pi/2)] + [\widehat{\psi}(-\epsilon) - \widehat{\psi}(\epsilon)], \quad \epsilon > 0. \end{aligned}$$

Since $\widehat{\psi}(y)$ is $\pi$-periodic, $\widehat{\psi}(\pi/2) - \widehat{\psi}(-\pi/2) = 0$. Therefore,

$$\begin{aligned} \Lambda(\epsilon) &= \widehat{\psi}(-\epsilon) - \widehat{\psi}(\epsilon) \\ &= \frac{(\sin\epsilon)^{\sigma+1}}{\sigma+1}\left(\frac{\sin\epsilon}{\sigma+2}[w'(\epsilon) - w'(-\epsilon)] - (\cos\epsilon)[w(\epsilon) + w(-\epsilon)]\right), \end{aligned}$$

and we choose to express this as

$$\Lambda(\epsilon) = \epsilon^{\sigma+2}A_1(\epsilon) + \epsilon^{\sigma+1}A_2(\epsilon),$$

with

$$A_1(\epsilon) = \frac{1}{(\sigma+1)(\sigma+2)}\left(\frac{\sin\epsilon}{\epsilon}\right)^{\sigma+2}[w'(\epsilon) - w'(-\epsilon)],$$
$$A_2(\epsilon) = -\frac{1}{\sigma+1}\left(\frac{\sin\epsilon}{\epsilon}\right)^{\sigma+1}(\cos\epsilon)[w(\epsilon) + w(-\epsilon)].$$

Clearly, in a neighborhood of $\epsilon = 0$, $A_1 \in C^{\widehat{m}-1}$ and $A_2 \in C^{\widehat{m}}$ because $w \in C^{\widehat{m}}$. Expanding $A_1(\epsilon)$ and $A_2(\epsilon)$ in their Maclaurin series with remainders, we obtain

$$\begin{aligned}\Lambda(\epsilon) &= \epsilon^{\sigma+2}\left[\sum_{i=0}^{\widehat{m}-2}\alpha_i\epsilon^i + O(\epsilon^{\widehat{m}-1})\right] + \epsilon^{\sigma+1}\left[\sum_{i=0}^{\widehat{m}-1}\beta_i\epsilon^i + O(\epsilon^{\widehat{m}})\right] \quad \text{as } \epsilon \to 0+,\\ &= \sum_{i=0}^{\widehat{m}-1}\gamma_i\epsilon^{\sigma+1+i} + O(\epsilon^{\sigma+1+\widehat{m}}) \quad \text{as } \epsilon \to 0+,\end{aligned}$$

with appropriate constants $\alpha_i$, $\beta_i$, and $\gamma_i$. We now observe that, because $\sigma = -(m+\delta)$, $0 < \delta < 1$, each of the terms involving $\epsilon^{\sigma+1+i}$, $i = 0, 1, \ldots, \widehat{m}-1$, tends either to infinity or to zero as $\epsilon \to 0+$; therefore, we discard all of them. The remainder term $O(\epsilon^{\sigma+1+\widehat{m}})$ tends to zero since $\sigma + 1 + \widehat{m} \geq 1 - \delta > 0$. We have thus proved that $M = 0$. Therefore, (2.16) becomes

$$\widehat{H}_\sigma(w) = \frac{\sigma+2}{\sigma+1}\widehat{H}_{\sigma+2}(w) + \frac{1}{(\sigma+1)(\sigma+2)}\widehat{H}_{\sigma+2}(w''), \tag{2.20}$$

which, by (2.8)–(2.9), gives (2.4)–(2.5). This completes the proof of the theorem. ■

Using Theorem 2.1, we now tackle the task of regularizing the HFP integrals described in (1.1). The point here is that the two HFP integrals $H_\sigma(t; \widetilde{u}_0)$ [with $\widetilde{u}_0(x) \equiv u(x)$] and $H_{\sigma+2}(t; \widetilde{u}_1)$ in (2.4) differ essentially in the strengths of the singularities at $x = t$ of their respective integrands; the former has a singularity of the form $|x-t|^\sigma$, while the latter has a (weaker) singularity of the form $|x-t|^{\sigma+2}$. They also differ in the smoothness properties of the respective $\widetilde{u}_i(x)$; $\widetilde{u}_0 \in C^{\widehat{m}}(\mathbb{R})$ while $\widetilde{u}_1 \in C^{\widehat{m}-2}(\mathbb{R})$. These facts enable us to apply Theorem 2.1 to $H_{\sigma+2}(t; \widetilde{u}_1)$ in (2.4)–(2.5), the end result being

$$H_{\sigma+2}(t; \widetilde{u}_1) = H_{\sigma+4}(t; \widetilde{u}_2),$$

where

$$\widetilde{u}_2(x) = \frac{\sigma+4}{\sigma+3}\widetilde{u}_1(x) + \frac{\theta^2}{(\sigma+3)(\sigma+4)}\widetilde{u}_1''(x).$$

Clearly, with $\widetilde{u}_1(x)$ as in (2.5), $\widetilde{u}_2(x)$ has the following structure:

$$\widetilde{u}_2(x) = \sum_{k=0}^{2}\beta_{\sigma,2,k}\theta^{2k}u^{(2k)}(x),$$

where

$$\beta_{\sigma,2,0} = \frac{(\sigma+2)(\sigma+4)}{(\sigma+1)(\sigma+3)},$$
$$\beta_{\sigma,2,1} = \frac{(\sigma+2)^2 + (\sigma+4)^2}{(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)},$$
$$\beta_{\sigma,2,2} = \frac{1}{(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)}.$$

Repeating this process enough times depending on $\sigma$, we arrive at a *regular* integral representing $H_\sigma(t; u)$, as described in the next theorem that can be proved by induction.

**Theorem 2.2.** *With $\sigma = -(m+\delta)$, $m \in \{1, 2, 3, \ldots\}$ and $0 < \delta < 1$, let*

$$r = \left\lfloor\frac{m+1}{2}\right\rfloor \quad \Rightarrow \quad m = \begin{cases}2r & \text{if } m \text{ even}\\ 2r-1 & \text{if } m \text{ odd}\end{cases} \quad \Rightarrow \quad \widehat{m} = 2r. \tag{2.21}$$

*Then, provided $u \in C^{2r}(\mathbb{R})$, there holds*

$$H_\sigma(t; u) = H_{\sigma+2r}(t; U_\sigma), \quad U_\sigma(x) = \widetilde{u}_r(x) = \sum_{k=0}^{r}\beta_{\sigma,r,k}\theta^{2k}u^{(2k)}(x), \tag{2.22}$$

*with appropriate constants $\beta_{\sigma,r,k}$ that depend only on $\sigma$, but not on $u(x)$ and $T$. In addition,*

$$\sigma+2r=\begin{cases}-\delta & \text{if } m \text{ even},\\ 1-\delta & \text{if } m \text{ odd},\end{cases} \quad\Rightarrow\quad \sigma+2r>-1, \tag{2.23}$$

*and, therefore, $H_{\sigma+2r}(t;\widetilde{u}_r)$ is a regular integral. Note that both $m$ and $r$ are determined uniquely by $\sigma$ as in*

$$m=\lfloor|\sigma|\rfloor \quad \text{and} \quad r=\left\lfloor\frac{\lfloor|\sigma|\rfloor+1}{2}\right\rfloor.$$

**Remark.** Note that the condition $u\in C^{2r}(\mathbb{R})$ already implies that $u\in C^{\widehat{m}}(\mathbb{R})$. This enables us to apply Theorem 2.1 $r$ times. Thus, (2.22) is obtained by applying Theorem 2.1 $r$ times, as follows:

$$H_\sigma(t;\widetilde{u}_0)=H_{\sigma+2}(t;\widetilde{u}_1)=H_{\sigma+4}(t;\widetilde{u}_2)=\cdots=H_{\sigma+2r-2}(t;\widetilde{u}_{r-1})=H_{\sigma+2r}(t;\widetilde{u}_r).$$

## 3. Construction of $H_\sigma(t;u)$ via Fourier series

### 3.1. Preliminaries

Going back to (1.1), we realize that, provided $u(x)$ is sufficiently differentiable on $\mathbb{R}$, $H_\sigma(t;u)$ is a $T$-periodic function of $t$. This prompts us to study its Fourier series $\sum_{q=-\infty}^{\infty}h_{\sigma,q}e_q(t)$ in the interval $[0,T]$, where

$$e_q(x)\equiv\exp(\mathrm{i}2q\pi x/T),\quad q=0,\pm1,\pm2,\ldots. \tag{3.1}$$

As will become clear shortly, working with the functions $\exp(\mathrm{i}2q\pi x/T)$ is much more convenient than working with $\sin(2q\pi x/T)$ and $\cos(2q\pi x/T)$.

We now present a complete study of $H_\sigma(t;e_q)$ for all $q$.

**Theorem 3.1.**

1. *For all $\sigma<-1$, $\sigma\notin\mathbb{Z}$, and for all $q$, there exist constants $M_{\sigma,q}$ independent of $t$, such that*

$$H_\sigma(t;e_q)=M_{\sigma,q}e_q(t),\quad q=0,\pm1,\pm2,\ldots, \tag{3.2}$$

*with*

$$M_{\sigma,q}=(-1)^q\frac{T}{2^\sigma}\frac{\Gamma(\sigma+1)}{\Gamma(\sigma/2+1+q)\Gamma(\sigma/2+1-q)}=M_{\sigma,-q}. \tag{3.3}$$

2. *In addition, for $q=0,1,2,\ldots$, the $M_{\sigma,q}$ can be computed recursively as follows:*

$$M_{\sigma,0}=\frac{T}{2^\sigma}\frac{\Gamma(\sigma+1)}{[\Gamma(\sigma/2+1)]^2};\quad M_{\sigma,q+1}=\frac{q-\sigma/2}{q+1+\sigma/2}M_{\sigma,q},\quad q=0,1,2,\ldots. \tag{3.4}$$

*Thus, knowledge of only $\Gamma(\sigma+1)$ and $\Gamma(\sigma/2+1)$ suffices for computing all of the $M_{\sigma,q}$.*

3. *Finally, the sequence $\{|M_{\sigma,q}|\}_{q=0}^{\infty}$ is monotonically increasing.*

**Proof.** We start by making the change of variable of integration $y=\pi(x-t)/T$ in $H_\sigma(t;e_q)$ and proceed as in (2.6). We observe that with $u(x)=e_q(x)$, we have $w(y)=\widehat{e}_q(y)e_q(t)$, where $\widehat{e}_q(y)=e^{\mathrm{i}2qy}$ and is $\pi$-periodic. Hence, by (2.6)–(2.8),

$$H_\sigma(t;e_q)=\left[\theta\widehat{H}_\sigma(\widehat{e}_q)\right]e_q(t)\quad\Rightarrow\quad M_{\sigma,q}=\theta\widehat{H}_\sigma(\widehat{e}_q). \tag{3.5}$$

We have thus proved (3.2).

We now apply (2.20) to $\widehat{H}_\sigma(\widehat{e}_q)$ and obtain

$$\begin{aligned}\widehat{H}_\sigma(\widehat{e}_q)&=\frac{\sigma+2}{\sigma+1}\widehat{H}_{\sigma+2}(\widehat{e}_q)+\frac{1}{(\sigma+1)(\sigma+2)}\widehat{H}_{\sigma+2}(\widehat{e}''_q)\\&=\frac{\sigma+2}{\sigma+1}\widehat{H}_{\sigma+2}(\widehat{e}_q)+\frac{(\mathrm{i}2q)^2}{(\sigma+1)(\sigma+2)}\widehat{H}_{\sigma+2}(\widehat{e}_q)\\&=\frac{(\sigma+2+2q)(\sigma+2-2q)}{(\sigma+1)(\sigma+2)}\widehat{H}_{\sigma+2}(\widehat{e}_q)\\&=4\frac{(\sigma/2+1+q)(\sigma/2+1-q)}{(\sigma+1)(\sigma+2)}\widehat{H}_{\sigma+2}(\widehat{e}_q).\end{aligned} \tag{3.6}$$

We next analyze the integral representation of $\widehat{H}_\sigma(\widehat{e}_q)$. By (2.7), and by the fact that $|\sin y|^\sigma\sin(2qy)$ is odd,

$$\neq\!\!\!\!\!\!\int_{-\pi/2}^{\pi/2}|\sin y|^\sigma\sin(2qy)\,dy=0=\neq\!\!\!\!\!\!\int_0^\pi(\sin y)^\sigma\sin(2qy)\,dy=0,$$

and, therefore,

$$\widehat{H}_\sigma(\widehat{e}_q) = \,=\!\!\!\!\!\!\int_{-\pi/2}^{\pi/2} |\sin y|^\sigma \cos(2qy)\,dy = \,=\!\!\!\!\!\!\int_0^\pi (\sin y)^\sigma \cos(2qy)\,dy.$$

By [16, §3.631(8)], for the *regular* integral $\int_0^\pi (\sin y)^c \cos(2qy)\,dy$, with $c > -1$, we have

$$\int_0^\pi (\sin y)^c \cos(2qy)\,dy = (-1)^q \frac{\pi}{2^c(c+1)} \frac{1}{B(c/2+1+q, c/2+1-q)},$$

where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{(Euler Beta function).}$$

For convenience, we rewrite this as

$$\int_0^\pi (\sin y)^c \cos(2qy)\,dy = (-1)^q \frac{\pi}{2^c} \frac{\Gamma(c+1)}{\Gamma(c/2+1+q)\Gamma(c/2+1-q)} \equiv R(c,q). \tag{3.7}$$

Here we have invoked $\Gamma(c+2) = (c+1)\Gamma(c+1)$. Note that even though the integral $\int_0^\pi (\sin y)^c \cos(2qy)\,dy$ is not regular, hence not defined, when $c < -1$, the quantity $R(c,q)$ is defined for *all* $c \neq -1, -2, \ldots$. [$R(c,q)$ is not defined for $c = -1, -2, \ldots$, because the factor $\Gamma(c+1)$ has poles at these values of $c$.]

Comparing (3.3) with (3.7), we see that what we need to show is that $M_{\sigma,q} = \theta R(\sigma, q)$. We prove, by induction on $j$, that (3.3) is true for $M_{\sigma+2r-2j,q}$, $j = 0, 1, 2, \ldots$. First, because $\sigma + 2r > -1$ by Theorem 2.2, hence the integral representation of $M_{\sigma+2r,q}$ is *regular*, we realize that $M_{\sigma+2r,q} = \theta R(\sigma+2r, q)$, hence (3.3) is true for $M_{\sigma+2r,q}$. We have thus shown the validity of (3.3) for $j = 0$. To complete the proof, it is sufficient to assume that the assertion for $j = r-1$ is correct and show that it is correct also for $j = r$. This amounts to showing that if (3.3) is true for $M_{\sigma+2,q}$ then it is true for $M_{\sigma,q}$. For all this, it is sufficient to work with the HFP integrals $\widehat{H}_\sigma(\widehat{e}_q)$ and $\widehat{H}_{\sigma+2}(\widehat{e}_q)$. Thus, we assume that (3.3) is valid for $\widehat{H}_{\sigma+2}(\widehat{e}_q)$, and we have

$$\widehat{H}_{\sigma+2}(\widehat{e}_q) = (-1)^q \frac{\pi}{2^{\sigma+2}} \frac{\Gamma(\sigma+3)}{\Gamma(\sigma/2+2+q)\Gamma(\sigma/2+2-q)}.$$

By repeated use of $\Gamma(z+1) = z\Gamma(z)$, this can be re-expressed as

$$\widehat{H}_{\sigma+2}(\widehat{e}_q) = (-1)^q \frac{\pi}{2^{\sigma+2}} \frac{(\sigma+2)(\sigma+1)\Gamma(\sigma+1)}{[(\sigma/2+1+q)\Gamma(\sigma/2+1+q)][(\sigma/2+1-q)\Gamma(\sigma/2+1-q)]}.$$

Substituting this into (3.6), and multiplying by $\theta$, we obtain (3.3). This completes the proof of part 1.

The proof of (3.4) in part 2 follows directly from (3.3).

The proof of part 3 follows from Eq. (3.4) and from the fact that

$$\left|\frac{M_{\sigma,q+1}}{M_{\sigma,q}}\right| = \left|\frac{q-\sigma/2}{q+1+\sigma/2}\right| > 1, \quad q = 0, 1, 2, \ldots, \quad \text{because } \sigma < -1.$$

We leave the details to the reader. ■

We next study the behavior of the $M_{\sigma,q}$ as $q \to \pm\infty$.

**Theorem 3.2.** *The $M_{\sigma,q}$ satisfy the asymptotic equality*

$$M_{\sigma,q} \sim K|q|^{-\sigma-1} \quad \text{as } q \to \pm\infty, \quad K = -\frac{T}{2^\sigma} \frac{\Gamma(\sigma+1)}{\Gamma(-\sigma/2-1)\Gamma(\sigma/2+2)}.$$

*Thus, when $\sigma = -(m+\delta)$ with $m = 1, 2, \ldots$, and $0 < \delta < 1$,*

$$M_{\sigma,q} = O(|q|^{m+\delta-1}) \quad \text{as } q \to \pm\infty; \quad m-1 < m+\delta-1 < m.$$

*Therefore, when $\sigma < -1$, $|M_{\sigma,q}| \to \infty$ as $q \to \pm\infty$.*

**Proof.** We start with the representation of $M_{\sigma,q}$ with $q > 0$, as given in (3.3). Letting $\sigma/2 + 1 = a$ for convenience, we need to study the product $\Gamma(a+q)\Gamma(a-q)$. Now

$$\Gamma(a-q) = (-1)^{q-1} \frac{\Gamma(-a)\Gamma(1+a)}{\Gamma(q+1-a)}.$$

Therefore,

$$M_{\sigma,q} = -\frac{T}{2^\sigma} \frac{\Gamma(\sigma+1)}{\Gamma(-a)\Gamma(1+a)} \frac{\Gamma(q+1-a)}{\Gamma(q+a)} \sim Kq^{1-2a} \quad \text{as } q \to \infty.$$

Here we have used the fact that $\Gamma(x+c) \sim \Gamma(x)x^c$ as $x \to \infty$. ■

We will make use of Theorem 3.2 when studying the convergence properties of our numerical quadrature formulas. In the next theorem, we show the connection between $M_{\sigma,q}$ and the $\beta_{\sigma,r,k}$ that we introduced in Theorem 2.2.

**Theorem 3.3.** *The scalars $\beta_{\sigma,r,k}$ in (2.22) are related to the $M_{\sigma,q}$ as in*

$$M_{\sigma,q} = \left[\sum_{k=0}^{r}(-1)^k \beta_{\sigma,r,k}\,(2q)^{2k}\right] M_{\sigma+2r,q}. \tag{3.8}$$

**Proof.** By Theorem 2.2, and the fact that $e_q^{(s)}(x) = (\mathrm{i}2q\pi/T)^s e_q(x)$, we have

$$\begin{aligned} H_\sigma(t; e_q) &= H_{\sigma+2r}\left(t; \sum_{k=0}^{r} \beta_{\sigma,r,k}\theta^{2k} e_q^{(2k)}\right) \\ &= H_{\sigma+2r}\left(t; \left[\sum_{k=0}^{r}(-1)^k \beta_{\sigma,r,k}(2q)^{2k}\right] e_q\right) \\ &= \left[\sum_{k=0}^{r}(-1)^k \beta_{\sigma,r,k}(2q)^{2k}\right] H_{\sigma+2r}(t; e_q). \end{aligned}$$

The result in (3.8) follows by recalling that

$$H_\sigma(t; e_q) = M_{\sigma,q} e_q(t) \quad \text{and} \quad H_{\sigma+2r}(t; e_q) = M_{\sigma+2r,q} e_q(t)$$

and that $e_q(t) \neq 0$ for all $t$. ∎

*3.2. Fourier series for $H_{\sigma+2r}(t; u^{(s)})$, $s = 0, 1, \ldots$*

In the sequel, we will be dealing with the Fourier series of the function $u(x)$, namely,

$$u(x) \sim \sum_{q=-\infty}^{\infty} c_q e^{\mathrm{i}2q\pi x/T}, \quad c_q = \frac{1}{T}\int_0^T u(x) e^{-\mathrm{i}2q\pi x/T}\,dx, \tag{3.9}$$

which converges [to $u(x)$] absolutely and uniformly provided $u(x)$ is $T$-periodic and differentiable on $\mathbb{R}$. We will also consider the Fourier series of $u^{(s)}(x)$, $s = 1, 2, \ldots$, namely,

$$u^{(s)}(x) \sim \sum_{q=-\infty}^{\infty} c_q^{(s)} e^{\mathrm{i}2q\pi x/T}, \quad c_q^{(s)} = \frac{1}{T}\int_0^T u^{(s)}(x) e^{-\mathrm{i}2q\pi x/T}\,dx = (\mathrm{i}2q/\theta)^s c_q. \tag{3.10}$$

We will deal with functions $u(x)$ in three different classes:

1. $u(x)$ is in the Hölder class $C^{M+1,\alpha}(\mathbb{R})$, $0 < \alpha \leq 1$. Thus $u^{(s)}(x)$, $s = 0, 1, \ldots, M$, are all continuous and $T$-periodic in $\mathbb{R}$, and $u^{(M+1)}(x)$ is in the Hölder class $C^{0,\alpha}(0, T)$, that is, $|u^{(M+1)}(x) - u^{(M+1)}(y)| \leq C|x - y|^\alpha$ for all $x, y \in [0, T]$ and for some constant $C > 0$. Then the Fourier coefficients $c_q$ of $u(x)$ are such that, for all $s = 0, 1, \ldots, M$,
$$c_q^{(s)} = O(|q|^{-M-\alpha-1+s}) \quad \text{as } q \to \pm\infty. \tag{3.11}$$
2. $u(x)$ is in $C^\infty(\mathbb{R})$. We now have for all $s = 0, 1, \ldots$,
$$c_q^{(s)} = O(|q|^{-\lambda}) \quad \text{as } q \to \pm\infty \quad \forall\, \lambda > 0. \tag{3.12}$$
3. As a function of the complex variable $z$, $u(z)$ is analytic in the infinite strip $D_\rho$,
$$D_\rho = \{z \in \mathbb{C} : |\mathbf{Im}\, z| < \rho\}. \tag{3.13}$$
In this case, we have for all $s = 0, 1, \ldots$,
$$|c_q^{(s)}| = O(e^{-2|q|\pi\widetilde{\rho}/T}) \quad \text{as } q \to \pm\infty \quad \forall\, \widetilde{\rho} \in (0, \rho), \tag{3.14}$$
Of course, now $u \in C^\infty(\mathbb{R})$ as well, but the result in (3.14) is much stronger and more informative than that in (3.12).

We now construct $H_{\sigma+2r}(t; u^{(s)})$ in terms of the Fourier series representation of $u^{(s)}(x)$, $s = 0, 1, \ldots, M$.

We begin with $u^{(0)}(x) = u(x)$. By the fact that $|\sin(\frac{\pi(x-t)}{T})|^{\sigma+2r}$ is absolutely integrable everywhere and because the Fourier series of $u(x)$ converges to $u(x)$ absolutely and uniformly everywhere, there holds

$$H_{\sigma+2r}(t; u) = H_{\sigma+2r}\left(t; \sum_{q=-\infty}^{\infty} c_q e_q\right) = \sum_{q=-\infty}^{\infty} c_q H_{\sigma+2r}(t; e_q),$$

which, by (3.2), becomes

$$H_{\sigma+2r}(t;u)=\sum_{q=-\infty}^{\infty}c_qM_{\sigma+2r,q}e_q(t). \tag{3.15}$$

Recalling also that $c_q^{(s)}=(\mathrm{i}2q/\theta)^s c_q$ for $q\neq 0$, with $s=1,2,\ldots$, we similarly have

$$H_{\sigma+2r}(t;u^{(s)})=\sum_{q=-\infty}^{\infty}c_q^{(s)}M_{\sigma+2r,q}e_q(t)=\sum_{q=-\infty}^{\infty}(\mathrm{i}2q/\theta)^s c_qM_{\sigma+2r,q}e_q(t). \tag{3.16}$$

Clearly, the right-hand side of (3.15) is the Fourier series of $H_{\sigma+2r}(t;u)$. Similarly, the right-hand side of (3.16) is the Fourier series of $H_{\sigma+2r}(t;u^{(s)})$ for $s=1,2,\ldots$.

### 3.3. Fourier series for $H_\sigma(t;u)$

Following the developments above, we now proceed to the construction of the Fourier series of $H_\sigma(t;u)$. We assume that $u(x)$ is as in the preceding subsection.

**Theorem 3.4.** *$H_\sigma(t;u)$ has the following Fourier series representation that converges absolutely and uniformly:*

$$H_\sigma(t;u)=\sum_{q=-\infty}^{\infty}c_qM_{\sigma,q}e_q(t). \tag{3.17}$$

**Proof.** We begin with the regularized $H_\sigma(t;u)$ as described in Theorem 2.2. We have

$$H_\sigma(t;u)=H_{\sigma+2r}(t;U_\sigma),\quad U_\sigma(x)=\sum_{k=0}^{r}\beta_{\sigma,r,k}\theta^{2k}u^{(2k)}(x). \tag{3.18}$$

Therefore,

$$H_\sigma(t;u)=\sum_{k=0}^{r}\beta_{\sigma,r,k}\theta^{2k}H_{\sigma+2r}(t;u^{(2k)}), \tag{3.19}$$

which, upon invoking (3.15)–(3.16), becomes

$$\begin{aligned}H_\sigma(t;u)&=\sum_{k=0}^{r}\beta_{\sigma,r,k}\theta^{2k}\sum_{q=-\infty}^{\infty}(\mathrm{i}2q/\theta)^{2k}c_qM_{\sigma+2r,q}e_q(t)\\&=\sum_{q=-\infty}^{\infty}c_q\left[\sum_{k=0}^{r}(-1)^k\beta_{\sigma,r,k}(2q)^{2k}\right]M_{\sigma+2r,q}e_q(t).\end{aligned}$$

Invoking now (3.8), we obtain (3.17). ■

**Remarks.**

1. One might think that the result in (3.17) (with $\sigma<-1$) should follow immediately by simply writing

$$\begin{aligned}H_\sigma(t;u)&=\,⨎_0^T\left|\sin\frac{\pi(x-t)}{T}\right|^\sigma\left(\sum_{q=-\infty}^{\infty}c_qe_q(x)\right)dx\\&=\sum_{q=-\infty}^{\infty}c_q⨎_0^T\left|\sin\frac{\pi(x-t)}{T}\right|^\sigma e_q(x)dx\\&=\sum_{q=-\infty}^{\infty}c_qM_{\sigma,q}e_q(t).\end{aligned}$$

   Despite the fact that the (infinite) series $\sum_{q=-\infty}^{\infty}c_qe_q(x)$ converges to $u(x)$ absolutely and uniformly on $\mathbb{R}$, the equality on the second line cannot be justified. The reason for this is that, when $\sigma<-1$, the integral on the first line does not exist in the regular sense as its integrand has a nonintegrable singularity at $x=t$ in $(0,T)$.
2. Since the Fourier expansion of $H_\sigma(t;u)$ in (3.17) has a surprisingly simple form, to remove any suspicion concerning the validity of (3.17), we have computed $H_\sigma(t;u)$ in the numerical examples of Section 6 by a completely different procedure that involves the application of the Richardson extrapolation process (see [17, Chapter 1]) to a

generalization of the Euler–Maclaurin expansion for HFP integrals derived by the author in [14]. Our computations confirm the validity of (3.17). We also mention that the application of Richardson extrapolation is much more expensive compared to the numerical quadrature method we describe in the next section. For more details on this subject, we refer the reader to the appendix to this work.

## 4. Numerical quadrature formula for $H_\sigma(t; u)$ via trigonometric interpolation

### 4.1. Development of the numerical quadrature formula

So far, we have seen that the $T$-periodic (divergent) HFP integral $H_\sigma(t; u)$ in (1.1) can be expressed as the *regular* integral

$$H_{\sigma+2r}(t; U_\sigma) = \int_0^T \left| \sin \frac{\pi(x-t)}{T} \right|^{\sigma+2r} U_\sigma(x)\,dx, \quad U_\sigma(x) = \sum_{k=0}^{r} \beta_{\sigma,r,k}\theta^{2k} u^{(2k)}(x).$$

We now present a numerical quadrature method for approximating $H_{\sigma+2r}(t; U_\sigma)$ without having to approximate the individual $H_{\sigma+2r}(t; u^{(2k)})$ that form $H_{\sigma+2r}(t; U_\sigma)$.[4]

We proceed as follows:

- We first approximate $u(x)$ on $[0, T]$ by a balanced trigonometric polynomial $\phi_n(x)$ of degree $n$ that interpolates $u(x)$ at $2n$ equidistant points $x_{n,0}, x_{n,1}, \ldots, x_{n,2n-1}$. As summarized in [1, Appendix], $\phi_n(x)$ is of the form

$$\phi_n(x) = {\sum_{q=-n}^{n}}'' \tilde{c}_{n,q} e_q(x), \quad \tilde{c}_{n,n} = \tilde{c}_{n,-n}, \tag{4.1}$$

  the double prime on the summation ${\sum''}_{q=-n}^{n}$ meaning that the terms with $q = \pm n$ are to be multiplied by 1/2, and

$$\phi_n(x_{n,k}) = u(x_{n,k}), \quad x_{n,k} = \frac{kT}{2n}, \quad k = 0, 1, \ldots, 2n-1, \tag{4.2}$$

  and

$$\tilde{c}_{n,q} = \frac{1}{2n} \sum_{k=0}^{2n-1} \overline{e_q(x_{n,k})} u(x_{n,k}) = \frac{1}{2n} \sum_{k=0}^{2n-1} e^{-\mathrm{i}qk\pi/n} u(x_{n,k}), \quad -n \le q \le n. \tag{4.3}$$

  Note that, for $\tilde{c}_{n,0}$, $\tilde{c}_{n,n}$, and $\tilde{c}_{n,-n}$, (4.3) gives

$$\tilde{c}_{n,0} = \frac{1}{2n} \sum_{k=0}^{2n-1} u(x_{n,k}); \quad \tilde{c}_{n,n} = \tilde{c}_{n,-n} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^k u(x_{n,k}). \tag{4.4}$$

- Next, we approximate $H_\sigma(t; u)$ by $H_\sigma(t; \phi_n)$. That is, our numerical quadrature formula $Q_{\sigma,n}(t; u)$ for $H_\sigma(t; u)$ is simply

$$Q_{\sigma,n}(t; u) = H_\sigma(t; \phi_n). \tag{4.5}$$

  Thus, because $\phi_n(x) = {\sum''}_{q=-n}^{n} \tilde{c}_{n,q} e_q(x)$ is a finite sum, we can immediately write

$$Q_{\sigma,n}(t; u) = H_\sigma\left(t; {\sum_{q=-n}^{n}}'' \tilde{c}_{n,q} e_q\right) = {\sum_{q=-n}^{n}}'' \tilde{c}_{n,q} H_\sigma(t; e_q), \tag{4.6}$$

  which, upon invoking (3.2), becomes

$$Q_{\sigma,n}(t; u) = {\sum_{q=-n}^{n}}'' \tilde{c}_{n,q} M_{\sigma,q} e_q(t). \tag{4.7}$$

  Clearly, in this form, $Q_{\sigma,n}(t; u)$ is very easy to compute once the $\tilde{c}_{n,q}$ have been computed.
  Substituting (4.3) into (4.7) and rearranging, we also obtain $Q_{\sigma,n}(t; u)$ as a trigonometric sum as follows:

$$Q_{\sigma,n}(t; u) = \frac{1}{2n} \sum_{k=0}^{2n-1} \left[ {\sum_{q=-n}^{n}}'' M_{\sigma,q} e_q(t - x_{n,k}) \right] u(x_{n,k}). \tag{4.8}$$

[4] The technical tools that are necessary for the developments of this section are provided with more detail in the appendix of the paper [1] by the author.

**Remarks.**

1. By (4.1)–(4.8), it is clear that the only input we need for computing $Q_{\sigma,n}(t;u)$, *for all* $t$, is the set $\{u(x_{n,0}), u(x_{n,1}), \ldots, u(x_{n,2n-1})\}$, which we use for computing the $\tilde{c}_{n,q}$; no derivative information from $u(x)$ is required.
2. By (4.5) and Theorem 2.2, we have that
$$Q_{\sigma,n}(t;u) = H_\sigma(t;\phi_n) = H_{\sigma+2r}(t;\Phi_{\sigma,n}),$$
where
$$\Phi_{\sigma,n}(x) = \sum_{j=0}^{r} \beta_{\sigma,r,j}\theta^{2j}\phi_n^{(2j)}(x).$$
This means that our numerical quadrature formula $Q_{\sigma,n}(t;u)$ replaces $u^{(2j)}(x)$ in the composition of $U_\sigma(x)$ resulting from $H_\sigma(t;u) = H_{\sigma+2r}(t;U_\sigma)$ by $\phi_n^{(2j)}(x)$. This takes place only *implicitly*, however, as is obvious from (4.7), since the $M_{\sigma,q}$ are readily available by Theorem 3.1.
3. Even though $\phi_n(x_{n,k}) = u(x_{n,k})$, we have only $\phi_n^{(2j)}(x_{nk}) \approx u^{(2j)}(x_{n,k})$, $j = 1, 2, \ldots, r$.

*4.2. Convergence of $Q_{\sigma,n}(t;u)$*

We now turn to the study of the convergence as $n \to \infty$ of $Q_{\sigma,n}(t;u)$. We begin by deriving upper bounds on the absolute errors $|Q_{\sigma,n}(t;u) - H_\sigma(t;u)|$ that we express in terms of the Fourier coefficients of $u(x)$.

We will make use of the following known facts, whose proofs can also be found in [1, Appendix], for example:

- As described above, trigonometric interpolation reproduces balanced trigonometric polynomials of degree at most $n$ in the following sense: if $u(x) = \sum''^{\,n}_{q=-n} c_q e_q(x)$ with $c_n = c_{-n}$, then $\phi_n(x) \equiv u(x)$.
- The $\tilde{c}_{n,q}$ are related to the $c_q$ as in
$$\tilde{c}_{n,p} = c_p + \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}. \tag{4.9}$$
- In case the infinite series $\sum_{q=-\infty}^{\infty} c_q$ converges absolutely, there holds
$$\sum_{|p|\le n}{}'' \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}| = \sum_{|q|\ge n}{}'' |c_q|. \tag{4.10}$$
By the notation $\sum''_{|p|\le n}$ and $\sum''_{|q|\ge n}$, we mean that the terms with $p = \pm n$ and $q = \pm n$, respectively, are to be multiplied by $1/2$.

**Theorem 4.1.** *The absolute error $|Q_{\sigma,n}(t;u) - H_\sigma(t;u)|$ can be bounded as follows:*
$$|Q_{\sigma,n}(t;u) - H_\sigma(t;u)| \le \sum_{|q|\ge n}{}'' |c_q|(|M_{\sigma,n}| + |M_{\sigma,q}|). \tag{4.11}$$
*Note that this bound is independent of $t$.*

**Proof.** By (3.17), (4.9), and (4.7), we have
$$\begin{aligned} Q_{\sigma,n}(t;u) - H_\sigma(t;u) &= \sum_{p=-n}^{n}{}'' \tilde{c}_{n,p}M_{\sigma,p}e_p(t) - \sum_{q=-\infty}^{\infty} c_q M_{\sigma,q}e_q(t) \\ &= \sum_{|p|\le n}{}'' \Bigg(c_p + \sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\Bigg)M_{\sigma,p}e_p(t) - \sum_{q=-\infty}^{\infty} c_q M_{\sigma,q}e_q(t) \\ &= \sum_{|p|\le n}{}'' \Bigg(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} c_{p+2n\tau}\Bigg)M_{\sigma,p}e_p(t) - \sum_{|q|\ge n}{}'' c_q M_{\sigma,q}e_q(t), \end{aligned}$$
which, upon taking moduli, gives
$$|Q_{\sigma,n}(t;u) - H_\sigma(t;u)| \le \sum_{|p|\le n}{}'' \Bigg(\sum_{\substack{\tau=-\infty\\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Bigg)|M_{\sigma,p}| + \sum_{|q|\ge n}{}'' |c_q||M_{\sigma,q}|. \tag{4.12}$$

To continue, we make use of the fact that, by Theorem Theorem 3.1, the sequence $\{|M_{\sigma,q}|\}_{q=0}^{\infty}$ is monotonically increasing, which implies that $\max\{|M_{\sigma,q}|\}_{q=0}^{n} = |M_{\sigma,n}|$. Therefore, (4.12) becomes

$$|Q_{\sigma,n}(t;u) - H_{\sigma}(t;u)| \le \Bigg(\sideset{}{''}\sum_{|p|\le n} \sum_{\substack{\tau=-\infty \\ \tau\neq 0}}^{\infty} |c_{p+2n\tau}|\Bigg)|M_{\sigma,n}| + \sideset{}{''}\sum_{|q|\ge n} |c_q||M_{\sigma,q}|$$
$$= \Bigg(\sideset{}{''}\sum_{|q|\ge n} |c_q|\Bigg)|M_{\sigma,n}| + \sideset{}{''}\sum_{|q|\ge n} |c_q||M_{\sigma,q}| \quad \text{by (4.10)},$$

which gives (4.11). ■

The next theorem provides the rates at which $Q_{\sigma,n}(t;u)$ converges to $H_{\sigma}(t;u)$ as $n \to \infty$ for all $t \in \mathbb{R}$. It follows from Theorem 4.1 and from (3.11)–(3.14) about the rates of growth of the $c_q$ as $q \to \pm\infty$.

**Theorem 4.2.**

1. *If $u(x)$ is $T$-periodic and in the Hölder class $C^{M+1,\alpha}(0,T)$, and if $M+\alpha+1 > -\sigma$, then*

$$|Q_{\sigma,n}(t;u) - H_{\sigma}(t;u)| = O(n^{-M-\alpha-\sigma-1}) \quad \text{as } n \to \infty. \tag{4.13}$$

2. *If $u(x)$ is $T$-periodic and in $C^{\infty}(\mathbb{R})$, then*

$$|Q_{\sigma,n}(t;u) - H_{\sigma}(t;u)| = o(n^{-\mu}) \quad \text{as } n \to \infty \quad \forall \mu > 0, \tag{4.14}$$

   *that is, $Q_{\sigma,n}(t;u)$ converges spectrally in $n$.*

3. *In case $u(z)$ is also $T$-periodic and analytic in an infinite strip $D_{\rho}$ of the complex $z$-plane, with $D_{\rho}$ as in (3.13), then*

$$|Q_{\sigma,n}(t;u) - H_{\sigma}(t;u)| = O(e^{-2n\pi\widetilde{\rho}/T}) \quad \text{as } n \to \infty \quad \forall \widetilde{\rho} \in (0,\rho), \tag{4.15}$$

   *that is, $Q_{\sigma,n}(t;u)$ converges exponentially in $n$, thus better than spectrally.*

*All these results are valid uniformly in $t$.*

### 4.3. Exactness property of $Q_{\sigma,n}(t;u)$

We now state some theorems that concern the exactness properties of $Q_{\sigma,n}(t;u)$. They can be proved exactly as the corresponding theorems in [1, Theorems 5.3, 5.4].

**Theorem 4.3.** *The numerical quadrature formula $Q_{\sigma,n}(t;u)$ has the following exactness property:*

$$Q_{\sigma,n}(t;u) = H_{\sigma}(t;u) \quad \text{if } u(x) = \sideset{}{''}\sum_{q=-n}^{n} c_q e_q(x) \text{ with } c_n = c_{-n}.$$

*In particular,*

$$Q_{\sigma,n}(t;e_q) = H_{\sigma}(t;e_q) = M_{\sigma,q}e_q(t), \quad q = 0, \pm 1, \ldots, \pm(n-1),$$

$$Q_{\sigma,n}(t;e_n + e_{-n}) = H_{\sigma}(t;e_n + e_{-n}) = M_{\sigma,n}e_n(t) + M_{\sigma,-n}e_{-n}(t) = M_{\sigma,n}[e_n(t) + e_{-n}(t)].$$

*In words, $Q_{\sigma,n}(t;u)$ reproduces $H_{\sigma}(t;u)$ when $u(x)$ is a balanced trigonometric polynomial of degree at most $n$.*

Following Theorem 4.3, which provides $Q_{\sigma,n}(t;e_q)$ for $|q| \le n-1$, Theorem 4.4 below provides $Q_{\sigma,n}(t;e_q)$ for $|q| \ge n$.

**Theorem 4.4.** *Define the set of integers $\Gamma$ as $\Gamma = \{\pm n, \pm 3n, \pm 5n, \ldots\}$.*

1. *If $q \in \Gamma$, then $q = (2j+1)n$ for some integer $j$, and*

$$Q_{\sigma,n}(t;e_q) = Q_{\sigma,n}(t;e_{\pm n}) = \frac{1}{2}[M_{\sigma,n}e_n(t) + M_{\sigma,-n}e_{-n}(t)] = \frac{1}{2}M_{\sigma,n}[e_n(t) + e_{-n}(t)]. \tag{4.16}$$

2. *If $q \notin \Gamma$, then there exist unique integers $\tau$ and $s$, $|s| \le n-1$, such that $q = 2n\tau + s$, and*

$$Q_{\sigma,n}(t;e_q) = Q_{\sigma,n}(t;e_s) = M_{\sigma,s}e_s(t). \tag{4.17}$$

## 5. Treatment of regular singular integrals

So far, we have dealt with the divergent integrals

$$\int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx,\quad 0<t<T,\quad \sigma<-1,\quad \sigma\notin\mathbb{Z}.$$

We recall that these integrals are singular and do not exist in the regular sense. They do exist in the HFP sense and we developed numerical quadrature formulas to compute their HFPs.

We now turn to the treatment of the integrals

$$\int_0^T \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx,\quad 0<t<T,\quad \sigma>-1,\quad \sigma\notin\mathbb{Z},$$

which are still singular, but they exist in the regular sense hence also in the sense of HFP. This means that they need no regularization. As will become clear, the developments of the preceding sections of this work apply to this case with only some minor changes. We will continue to denote these integrals by $H_\sigma(t;u)$. The theoretical developments of Section 3 are the key to the following results.

**Theorem 5.1.**

1. *For all $\sigma>-1$, $\sigma\notin\mathbb{Z}$, and for all $q$, there exist constants $M_{\sigma,q}$ independent of $t$, such that*

$$H_\sigma(t;e_q)=M_{\sigma,q}e_q(t),\quad q=0,\pm1,\pm2,\ldots, \tag{5.1}$$

*with*

$$M_{\sigma,q}=(-1)^q\frac{T}{2^\sigma}\frac{\Gamma(\sigma+1)}{\Gamma(\sigma/2+1+q)\Gamma(\sigma/2+1-q)}=M_{\sigma,-q}. \tag{5.2}$$

2. *In addition, for $q=0,1,2,\ldots$, the $M_{\sigma,q}$ can be computed recursively as follows:*

$$M_{\sigma,0}=\frac{T}{2^\sigma}\frac{\Gamma(\sigma+1)}{[\Gamma(\sigma/2+1)]^2};\quad M_{\sigma,q+1}=\frac{q-\sigma/2}{q+1+\sigma/2}M_{\sigma,q},\quad q=0,1,2,\ldots. \tag{5.3}$$

*Thus, knowledge of only $\Gamma(\sigma+1)$ and $\Gamma(\sigma/2+1)$ suffices for computing all of the $M_{\sigma,q}$.*

3. *Finally, the sequence $\{|M_{\sigma,q}|\}_{q=0}^\infty$ is monotonically decreasing, and the $M_{\sigma,q}$ satisfy*

$$M_{\sigma,q}\sim K|q|^{-\sigma-1}\quad \text{as } q\to\pm\infty,\quad K=-\frac{T}{2^\sigma}\frac{\Gamma(\sigma+1)}{\Gamma(-\sigma/2-1)\Gamma(\sigma/2+2)}. \tag{5.4}$$

*Therefore, when $\sigma>-1$, $|M_{\sigma,q}|\to 0$ as $q\to\pm\infty$.*

**Proof.** The proof of (5.1) is exactly the same as that of (3.2). The proof of (5.2) follows directly from (3.7). The proof of the rest of the theorem is the same as the rest of the proofs of Theorems 3.1 and 3.2. ∎

The following theorem concerns the Fourier series representation of $H_\sigma(t;u)$. As before, we assume that $u(x)$ is either in the Hölder class $C^{M+1,\alpha}(\mathbb{R})$, $0<\alpha\le 1$, or is in $C^\infty(\mathbb{R})$, or, as a function of the complex variable $z$, $u(z)$ is analytic in the infinite strip $D_\rho$, with $D_\rho$ as in (3.13),

**Theorem 5.2.** *Let $u(x)$ have the Fourier series representation*

$$u(x)\sim\sum_{q=-\infty}^{\infty}c_qe_q(t),\quad c_q=\frac{1}{T}\int_0^T\overline{e_q(x)}u(x)\,dx.$$

*Then $H_\sigma(t;u)$ has the following Fourier series representation that converges absolutely and uniformly:*

$$H_\sigma(t;u)=\sum_{q=-\infty}^{\infty}c_qM_{\sigma,q}e_q(t). \tag{5.5}$$

**Proof.** The proof follows from the facts that (i) $|\sin\frac{\pi(x-t)}{T}|^\sigma$ is absolutely integrable on $[0,T]$ because $\sigma>-1$, and (ii) the Fourier series of $u(x)$ converges to $u(x)$ on $[0,T]$ absolutely and uniformly. ∎

In view of all this, we propose to approximate $H_\sigma(t;u)$ precisely as described in Section 4. That is, we approximate $u(x)$ on $[0,T]$ by the trigonometric interpolant $\phi_n(x)=\sum''^{\,n}_{q=-n}\tilde{c}_{n,q}e_q(x)$, and approximate $H_\sigma(t;u)$ by $Q_{\sigma,n}(t;u)=H_\sigma(t;\phi_n)$. Thus

$$Q_{\sigma,n}(t;u)=\sum_{q=-n}^{n}{}''\,\tilde{c}_{n,q}M_{\sigma,q}e_q(t). \tag{5.6}$$

We then have the following theorem that concerns the error $Q_{\sigma,n}(t;u) - H_\sigma(t;u)$ and the convergence of $Q_{\sigma,n}(t;u)$ as $n \to \infty$. Its proof is almost identical to that of Theorem 4.1, where we also take into account that $|M_{\sigma,0}| = \max\{|M_{\sigma,q}|\}_{q=0}^{\infty}$ and that $\lim_{q\to\pm\infty} M_{\sigma,q} = 0$.

**Theorem 5.3.** *Concerning $Q_{\sigma,n}(t;u)$, we have*

$$|Q_{\sigma,n}(t;u) - H_\sigma(t;u)| \leq \sum_{|q|\geq n}{}' |c_q|(|M_{\sigma,0}| + |M_{\sigma,q}|). \tag{5.7}$$

**Theorem 5.4.**

1. *If $u(x)$ is $T$-periodic and in the Hölder class $C^{M+1,\alpha}(0,T)$, then*
$$|Q_{\sigma,n}(t;u) - H_\sigma(t;u)| = O(n^{-M-\alpha}) \quad \text{as } n \to \infty. \tag{5.8}$$
2. *If $u(x)$ is $T$-periodic and in $C^\infty(\mathbb{R})$, then*
$$|Q_{\sigma,n}(t;u) - H_\sigma(t;u)| = o(n^{-\mu}) \quad \text{as } n \to \infty \quad \forall \mu > 0, \tag{5.9}$$
*that is, $Q_{\sigma,n}(t;u)$ converges spectrally in $n$.*
3. *In case $u(z)$ is also $T$-periodic and analytic in an infinite strip $D_\rho$ of the complex $z$-plane, with $D_\rho$ as in (3.13), then*
$$|Q_{\sigma,n}(t;u) - H_\sigma(t;u)| = O(e^{-2n\pi\tilde{\rho}/T}) \quad \text{as } n \to \infty \quad \forall \tilde{\rho} \in (0,\rho), \tag{5.10}$$
*that is, $Q_{\sigma,n}(t;u)$ converges exponentially in $n$, thus better than spectrally.*

*All these results are valid uniformly in $t$.*

## 6. Numerical examples

We now apply the numerical method we have just developed to the integrals $H_\sigma(t;u)$, with $T = 2\pi$ (hence $\theta = 2$) and

$$u(x) = \sum_{q=0}^{\infty} \eta^q \cos qx = \boldsymbol{Re}\frac{1}{1-\eta e^{ix}} = \frac{1-\eta\cos x}{1-2\eta\cos x + \eta^2}, \quad \eta \text{ real}, \quad 0 < \eta < 1. \tag{6.1}$$

Clearly, $u(x)$ is infinitely differentiable and $2\pi$-periodic on $\mathbb{R}$. Therefore, $T = 2\pi$ throughout. In addition, $u(x)$ can be continued to the complex $z$-plane, such that $u(z)$ is also $2\pi$-periodic and analytic in the infinite strip $D_\rho$ in (3.13) with $\rho = \log \eta^{-1}$. Finally, $H_\sigma(t;u)$ can be computed numerically by summing the Fourier series in (3.17) and (5.5) as follows: We first have

$$c_0 = 1; \quad c_q = \eta^{|q|}/2, \quad q \neq 0.$$

Next, by Theorem 4.2 with $\sigma < -1$, and by Theorem 5.2 with $\sigma > -1$, and also by the fact that $M_{\sigma,q} = M_{\sigma,-q}$ for all $q$, we have the following Fourier series for $H_\sigma(t;u)$:

$$H_\sigma(t;u) = M_{\sigma,0} + \sum_{q=1}^{\infty} \eta^q M_{\sigma,q} \cos qt \quad \forall \sigma \notin \mathbb{Z}. \tag{6.2}$$

Recalling that $M_{\sigma,q} = O(|q|^{-\sigma-1})$ as $q \to \pm\infty$, we realize that these Fourier series converge very quickly and enable us to compute $H_\sigma(t;u)$ easily, whether $\sigma < -1$ or $\sigma > -1$.

We have applied our quadrature formulas $Q_{\sigma,n}(t;u)$, as shown in (4.7) and (5.6), to the integrals $H_\sigma(t;u)$ with (i) $\sigma = 0.5, -0.5$ for regular integrals, and (ii) $\sigma = -1.5, -2.5, -3.5, -4.5$ for HFP integrals, using quadruple-precision arithmetic with roundoff unit $\mathbf{u} = 1.93 \times 10^{-34}$.

The results of our computations, with $t = 1$ in all cases, are shown in Tables 6.1–6.6. Note that because $u(z)$ is analytic in the infinite strip $D_\rho$ with $\rho = \log\eta^{-1}$, we have that the error $[Q_{\sigma,n}(t;u) - H_\sigma(t;u)]$ tends to zero as $n \to \infty$ exponentially in $n$ like $\eta^n$ by Theorems 4.2 and 5.4. Our numerical results confirm this amply for the different values of $\eta$. Finally, we recall that $Q_{\sigma,n}(t;u)$ requires only $2n$ evaluations of $u(x)$ and no evaluations of derivatives of $u(x)$.

## 7. Application of $Q_{\sigma,n}(t;\cdot)$ to general singular integrals and integral equations

### 7.1. Application of $Q_{\sigma,n}(t;\cdot)$ to singular integrals

So far, we have dealt with the HFP integrals $H_\sigma(t;u) = \int_0^T f(x,t)\,dx$, where $f(x,t)$ is $T$-periodic in $x$ and is expressed as

$$f(x,t) = \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)\,dx, \quad 0 < t < T, \quad \sigma \notin \mathbb{Z}, \tag{7.1}$$

**Table 6.1**
Numerical results for $Q_{\sigma,n}(t;u)$ with $\sigma = 0.5$, $t = 1$, and $u(x)$ as in (6.1). Here $E_n(\eta = c) = |Q_{\sigma,n}(t;u) - H_\sigma(t;u)|/|H_\sigma(t;u)|$ for $\eta = c$. [In this case, $H_\sigma(t;u)$ is defined in the regular sense.]

| $n$ | $E_n(\eta = 0.1)$ | $E_n(\eta = 0.2)$ | $E_n(\eta = 0.3)$ | $E_n(\eta = 0.4)$ | $E_n(\eta = 0.5)$ |
|---|---|---|---|---|---|
| 10 | 6.78D−14 | 1.46D−10 | 1.29D−08 | 3.13D−07 | 4.32D−06 |
| 20 | 3.64D−24 | 8.37D−18 | 4.49D−14 | 1.98D−11 | 2.19D−09 |
| 30 | 4.87D−34 | 4.98D−25 | 1.56D−19 | 1.24D−15 | 1.29D−12 |
| 40 | 4.87D−34 | 2.48D−32 | 4.51D−25 | 6.39D−20 | 6.25D−16 |
| 50 | 0.00D+00 | 0.00D+00 | 6.71D−31 | 1.72D−24 | 1.60D−19 |
| 60 | 4.87D−34 | 1.64D−34 | 3.31D−34 | 1.51D−28 | 1.25D−22 |
| 70 | 4.87D−34 | 3.28D−34 | 3.31D−34 | 3.17D−32 | 2.54D−25 |
| 80 | 0.00D+00 | 0.00D+00 | 3.31D−34 | 1.67D−34 | 2.63D−28 |
| 90 | 6.50D−34 | 0.00D+00 | 3.31D−34 | 3.34D−34 | 1.94D−31 |
| 100 | 6.50D−34 | 8.20D−34 | 3.31D−34 | 1.00D−33 | 0.00D+00 |
| 110 | 1.62D−34 | 1.64D−34 | 0.00D+00 | 1.33D−33 | 5.03D−34 |
| 120 | 3.25D−34 | 1.64D−34 | 6.62D−34 | 8.34D−34 | 3.36D−34 |

**Table 6.2**
Numerical results for $Q_{\sigma,n}(t;u)$ with $\sigma = -0.5$, $t = 1$, and $u(x)$ as in (6.1). Here $E_n(\eta = c) = |Q_{\sigma,n}(t;u) - H_\sigma(t;u)|/|H_\sigma(t;u)|$ for $\eta = c$. [In this case, $H_\sigma(t;u)$ is defined in the regular sense.]

| $n$ | $E_n(\eta = 0.1)$ | $E_n(\eta = 0.2)$ | $E_n(\eta = 0.3)$ | $E_n(\eta = 0.4)$ | $E_n(\eta = 0.5)$ |
|---|---|---|---|---|---|
| 10 | 1.11D−12 | 2.41D−09 | 2.18D−07 | 5.30D−06 | 6.14D−05 |
| 20 | 1.27D−22 | 2.88D−16 | 1.53D−12 | 6.72D−10 | 7.47D−08 |
| 30 | 1.08D−32 | 2.60D−23 | 7.97D−18 | 6.25D−14 | 6.50D−11 |
| 40 | 1.44D−34 | 1.73D−30 | 3.07D−23 | 4.29D−18 | 4.16D−14 |
| 50 | 5.78D−34 | 1.43D−34 | 5.72D−29 | 1.43D−22 | 1.30D−17 |
| 60 | 1.44D−34 | 2.85D−34 | 2.83D−34 | 1.56D−26 | 1.30D−20 |
| 70 | 1.44D−34 | 8.55D−34 | 4.24D−34 | 3.86D−30 | 3.02D−23 |
| 80 | 7.22D−34 | 4.28D−34 | 1.55D−33 | 1.83D−33 | 3.55D−26 |
| 90 | 7.22D−34 | 4.28D−34 | 2.83D−34 | 8.46D−34 | 2.94D−29 |
| 100 | 2.89D−34 | 4.28D−34 | 1.27D−33 | 9.87D−34 | 1.63D−32 |
| 110 | 1.16D−33 | 0.00D+00 | 4.24D−34 | 1.41D−34 | 1.27D−33 |
| 120 | 4.33D−34 | 1.43D−34 | 1.41D−34 | 1.41D−34 | 9.91D−34 |

**Table 6.3**
Numerical results for $Q_{\sigma,n}(t;u)$ with $\sigma = -1.5$, $t = 1$, and $u(x)$ as in (6.1). Here $E_n(\eta = c) = |Q_{\sigma,n}(t;u) - H_\sigma(t;u)|/|H_\sigma(t;u)|$ for $\eta = c$. [In this case, $H_\sigma(t;u)$ is defined in the HFP sense only.]

| $n$ | $E_n(\eta = 0.1)$ | $E_n(\eta = 0.2)$ | $E_n(\eta = 0.3)$ | $E_n(\eta = 0.4)$ | $E_n(\eta = 0.5)$ |
|---|---|---|---|---|---|
| 10 | 8.03D−11 | 1.73D−07 | 1.69D−05 | 5.05D−04 | 8.97D−03 |
| 20 | 1.97D−20 | 4.27D−14 | 2.37D−10 | 1.23D−07 | 2.01D−05 |
| 30 | 2.63D−30 | 5.81D−21 | 1.86D−15 | 1.71D−11 | 2.58D−08 |
| 40 | 1.07D−32 | 5.20D−28 | 9.54D−21 | 1.56D−15 | 2.18D−11 |
| 50 | 1.52D−32 | 1.26D−32 | 2.22D−26 | 6.38D−20 | 8.27D−15 |
| 60 | 2.09D−32 | 2.30D−32 | 1.40D−31 | 8.67D−24 | 1.06D−17 |
| 70 | 2.19D−31 | 2.12D−31 | 2.24D−31 | 2.48D−27 | 2.82D−20 |
| 80 | 2.08D−31 | 2.08D−31 | 2.25D−31 | 6.24D−31 | 3.77D−23 |
| 90 | 6.01D−32 | 6.07D−32 | 6.48D−32 | 7.58D−32 | 3.51D−26 |
| 100 | 1.96D−31 | 2.21D−31 | 2.68D−31 | 3.52D−31 | 2.09D−29 |
| 110 | 2.33D−31 | 2.24D−31 | 2.44D−31 | 3.19D−31 | 5.46D−31 |
| 120 | 1.29D−31 | 1.05D−31 | 8.95D−32 | 9.21D−32 | 1.26D−31 |

whether $\sigma < -1$ or $\sigma > -1$, $u(x)$ being $T$-periodic and sufficiently differentiable on $\mathbb{R}$. We also mentioned that the Cauchy-like transforms $J_\sigma(t;w)$ on the unit circle described in Section 1 are actually $H_\sigma(t;\widehat{w})$, where $\widehat{w}(x) = \mathrm{i}2^\sigma e^{\mathrm{i}x} w(e^{\mathrm{i}x})$. We now consider HFP integrals $⨍_0^T f(x,t)\,dx$ whose integrands $f(x,t)$ have algebraic singularities of the form $|x-t|^\sigma$, whether $\sigma < -1$ or $\sigma > -1$, and $\sigma \notin \mathbb{Z}$, but are not expressed necessarily as in (7.1).

**Example 7.1.** Consider the HFP integral $I_\sigma(t;f) = ⨍_0^T f(x,t)\,dx$, where $f(x,t)$ is $T$-periodic in $x \in \mathbb{R}$ and, for $x \in [0,T]$, it has the form

$$f(x,t) = g(x,t)|x-t|^\sigma, \quad 0 < t < T, \tag{7.2}$$

such that $g(x,t)$ is sufficiently differentiable as a function of $x$ on $[0,T]$. In addition, $t$ is being held fixed throughout. [Note that $g(x,t)$ is *not* $T$-periodic since $|x-t|^\sigma$ is not.]

**Table 6.4**
Numerical results for $Q_{\sigma,n}(t;u)$ with $\sigma = -2.5$, $t = 1$, and $u(x)$ as in (6.1). Here $E_n(\eta = c) = |Q_{\sigma,n}(t;u) - H_\sigma(t;u)|/|H_\sigma(t;u)|$ for $\eta = c$. [In this case, $H_\sigma(t;u)$ is defined in the HFP sense only.]

| $n$ | $E_n(\eta = 0.1)$ | $E_n(\eta = 0.2)$ | $E_n(\eta = 0.3)$ | $E_n(\eta = 0.4)$ | $E_n(\eta = 0.5)$ |
|---|---|---|---|---|---|
| 10 | 1.90D−09 | 3.84D−06 | 2.26D−04 | 3.38D−03 | 2.69D−02 |
| 20 | 9.88D−19 | 1.95D−12 | 6.30D−09 | 1.59D−06 | 1.13D−04 |
| 30 | 2.00D−28 | 4.01D−19 | 7.38D−14 | 3.28D−10 | 2.14D−07 |
| 40 | 1.50D−31 | 4.79D−26 | 5.06D−19 | 3.96D−14 | 2.40D−10 |
| 50 | 2.34D−30 | 2.12D−30 | 1.47D−24 | 2.00D−18 | 1.10D−13 |
| 60 | 6.46D−30 | 5.87D−30 | 9.47D−30 | 3.38D−22 | 1.82D−16 |
| 70 | 3.52D−29 | 3.07D−29 | 1.89D−29 | 1.11D−25 | 5.51D−19 |
| 80 | 2.77D−29 | 2.52D−29 | 1.53D−29 | 2.70D−29 | 8.38D−22 |
| 90 | 1.05D−29 | 9.78D−30 | 5.69D−30 | 3.25D−30 | 8.73D−25 |
| 100 | 3.36D−29 | 3.59D−29 | 2.59D−29 | 1.67D−29 | 5.72D−28 |
| 110 | 4.07D−29 | 3.53D−29 | 2.26D−29 | 1.48D−29 | 1.17D−29 |
| 120 | 1.85D−29 | 1.05D−29 | 3.85D−30 | 8.40D−31 | 8.88D−32 |

**Table 6.5**
Numerical results for $Q_{\sigma,n}(t;u)$ with $\sigma = -3.5$, $t = 1$, and $u(x)$ as in (6.1). Here $E_n(\eta = c) = |Q_{\sigma,n}(t;u) - H_\sigma(t;u)|/|H_\sigma(t;u)|$ for $\eta = c$. [In this case, $H_\sigma(t;u)$ is defined in the HFP sense only.]

| $n$ | $E_n(\eta = 0.1)$ | $E_n(\eta = 0.2)$ | $E_n(\eta = 0.3)$ | $E_n(\eta = 0.4)$ | $E_n(\eta = 0.5)$ |
|---|---|---|---|---|---|
| 10 | 1.34D−08 | 1.86D−05 | 9.27D−04 | 1.45D−02 | 1.38D−01 |
| 20 | 1.47D−17 | 1.93D−11 | 5.08D−08 | 1.31D−05 | 1.09D−03 |
| 30 | 4.52D−27 | 5.96D−18 | 8.91D−13 | 3.99D−09 | 3.05D−06 |
| 40 | 5.90D−30 | 9.53D−25 | 8.13D−18 | 6.40D−13 | 4.50D−09 |
| 50 | 9.73D−29 | 5.95D−29 | 2.95D−23 | 3.98D−17 | 2.51D−12 |
| 60 | 3.70D−28 | 2.18D−28 | 2.07D−28 | 8.35D−21 | 5.34D−15 |
| 70 | 1.70D−27 | 9.76D−28 | 4.86D−28 | 3.18D−24 | 1.85D−17 |
| 80 | 1.10D−27 | 6.64D−28 | 3.23D−28 | 7.78D−28 | 3.19D−20 |
| 90 | 4.67D−28 | 2.95D−28 | 1.35D−28 | 7.84D−29 | 3.72D−23 |
| 100 | 1.96D−27 | 1.41D−27 | 8.44D−28 | 5.54D−28 | 2.68D−26 |
| 110 | 2.45D−27 | 1.39D−27 | 7.35D−28 | 5.04D−28 | 4.85D−28 |
| 120 | 8.13D−28 | 1.56D−28 | 4.60D−29 | 9.87D−29 | 1.19D−28 |

**Table 6.6**
Numerical results for $Q_{\sigma,n}(t;u)$ with $\sigma = -4.5$, $t = 1$, and $u(x)$ as in (6.1). Here $E_n(\eta = c) = |Q_{\sigma,n}(t;u) - H_\sigma(t;u)|/|H_\sigma(t;u)|$ for $\eta = c$. [In this case, $H_\sigma(t;u)$ is defined in the HFP sense only.]

| $n$ | $E_n(\eta = 0.1)$ | $E_n(\eta = 0.2)$ | $E_n(\eta = 0.3)$ | $E_n(\eta = 0.4)$ | $E_n(\eta = 0.5)$ |
|---|---|---|---|---|---|
| 10 | 1.36D−07 | 2.76D−04 | 6.68D−03 | 1.25D−01 | 6.75D+00 |
| 20 | 3.15D−16 | 5.75D−10 | 7.13D−07 | 2.14D−04 | 1.00D−01 |
| 30 | 1.47D−25 | 2.67D−16 | 1.87D−11 | 9.74D−08 | 4.14D−04 |
| 40 | 5.47D−28 | 5.72D−23 | 2.27D−16 | 2.07D−11 | 8.08D−07 |
| 50 | 5.48D−27 | 4.68D−27 | 1.03D−21 | 1.58D−15 | 5.45D−10 |
| 60 | 2.61D−26 | 2.13D−26 | 8.26D−27 | 4.13D−19 | 1.49D−12 |
| 70 | 1.18D−25 | 9.35D−26 | 2.18D−26 | 1.82D−22 | 5.91D−15 |
| 80 | 6.26D−26 | 5.28D−26 | 1.19D−26 | 4.65D−26 | 1.16D−17 |
| 90 | 2.90D−26 | 2.62D−26 | 5.45D−27 | 3.71D−27 | 1.52D−20 |
| 100 | 1.78D−25 | 1.81D−25 | 5.09D−26 | 3.91D−26 | 1.21D−23 |
| 110 | 2.31D−25 | 1.82D−25 | 4.57D−26 | 3.73D−26 | 2.06D−25 |
| 120 | 5.30D−26 | 1.05D−26 | 1.26D−26 | 1.58D−26 | 9.35D−26 |

Let us define the function $u(x,t)$ via

$$g(x,t)|x-t|^\sigma = \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x,t).$$

Therefore,

$$u(x,t) = \left(\frac{T}{\pi}\right)^\sigma \frac{g(x,t)}{\left|\mathrm{sinc}\left(\frac{\pi(x-t)}{T}\right)\right|^\sigma} \quad \text{and} \quad u(t,t) = \left(\frac{T}{\pi}\right)^\sigma g(t,t),$$

where

$$\mathrm{sinc}(z) = \frac{\sin z}{z}$$

is the sinc function, which is defined and is easily computable everywhere and is positive for $z \in (-\pi, \pi)$, with $\mathrm{sinc}(0) = 1$. Clearly, $u(x, t)$ is $T$-periodic and has no singularity and is as differentiable as $g(x, t)$. In addition, it can be computed easily when $g(x, t)$ is available or can be computed easily.

**Example 7.2.** Next, we consider the HFP integral $I_\sigma(t; f) = \fint_0^T f(x, t)\, dx$, where $f(x, t)$ is $T$-periodic in $x \in \mathbb{R}$ and has the general form

$$f(x,t) = g(x,t)|\psi(x) - \psi(t)|^\sigma, \quad 0 < t < T, \tag{7.3}$$

such that $g(x, t)$ and $\psi(x)$ are sufficiently differentiable as functions of $x$ and are also $T$-periodic. In addition $\psi'(x) \neq 0$ on $[0, T]$.

Let us define the function $\psi[x, t]$ as follows:

$$\psi[x,t] = \begin{cases} \dfrac{\psi(x) - \psi(t)}{x - t} & \text{if } x \neq t, \\ \psi'(t) & \text{if } x = t. \end{cases}$$

Then, we can write

$$f(x,t) = \left| \sin \frac{\pi(x-t)}{T} \right|^\sigma u(x,t),$$

where

$$u(x,t) = \left(\frac{T}{\pi}\right)^\sigma \left| \frac{\psi[x,t]}{\mathrm{sinc}(\frac{\pi(x-t)}{T})} \right|^\sigma g(x,t) \quad \text{and} \quad u(t,t) = \left(\frac{T}{\pi}|\psi'(t)|\right)^\sigma g(t,t).$$

It is easy to verify that $u(x, t)$ is $T$-periodic and has no singularity and is as differentiable as $g(x, t)$. In addition, it can be computed easily when $g(x, t)$ is available or can be computed easily.

### 7.2. Application of $Q_{\sigma,n}(t; \cdot)$ to singular integral equations

Finally, the approach to the solution of linear integral equations with strongly singular kernels that we developed in [1, Section 7] can be applied here too. We now consider an integral equation of the form

$$\lambda w(t) + \fint_0^T G(x,t) w(x)\, dx = a(t), \quad t \in [0, T], \tag{7.4}$$

which is related to Example 7.2.

Here the constant $\lambda$ and the functions $G(x, t)$ and $a(x)$ are known; $w(x)$ is the unknown function, hence is the required solution to this equation. $G(x, t)$ is usually some sort of Green's function with an algebraic singularity of the form $|x - t|^\sigma$ when $x, t \in (0, T)$, with $\sigma < -1$ and $\sigma \notin \mathbb{Z}$, and $a(x)$ and $w(x)$ are $T$-periodic and sufficiently differentiable on $\mathbb{R}$.

Here we consider the case where (i) $G(x, t) = g(x, t)|\psi(x) - \psi(t)|^\sigma$, $g(x, t)$ being $T$-periodic in $x$ and $t$ and sufficiently differentiable on $\mathbb{R}$, and (ii) $\psi(x)$ is $T$-periodic and sufficiently differentiable on $\mathbb{R}$, as in Example 7.2. Additional conditions may have to be imposed on $G(x, t)$ and/or $a(x)$ to ensure uniqueness of solution; we will skip this issue below.

We now define the function $u(x, t)$ as the solution to the equation

$$G(x,t) w(x) = \left| \sin \frac{\pi(x-t)}{T} \right|^\sigma u(x,t).$$

After some simple manipulation, we obtain

$$u(x,t) = N(x,t) w(x), \quad N(x,t) = \left(\frac{T}{\pi}\right)^\sigma g(x,t) \left| \frac{\psi[x,t]}{\mathrm{sinc}(\frac{\pi(x-t)}{T})} \right|^\sigma.$$

Clearly, $N(x, t)$ is $T$-periodic and sufficiently differentiable on $\mathbb{R}$ and

$$N(t,t) = \left(\frac{T}{\pi}|\psi'(t)|\right)^\sigma g(t,t).$$

We now turn to the numerical solution of the integral equation in (7.4), which we can now write as

$$\lambda w(t) + \fint_0^T \left| \sin \frac{\pi(x-t)}{T} \right|^\sigma u(x,t)\, dx = a(t), \quad t \in [0, T],$$

hence also as

$$\lambda w(t) + H_\sigma(t; u(\cdot, t)) = a(t) \quad \Rightarrow \quad \lambda w(t) + H_\sigma(t; N(\cdot, t) w(\cdot)) = a(t), \quad t \in [0, T].$$

First, we set $x_{n,k} = kT/(2n)$, $k = 0, 1, \dots, 2n-1$, and replace $H_\sigma(t; N(\cdot,t)w(\cdot))$ by $Q_{\sigma,n}(t; N(\cdot,t)w(\cdot))$. Next, we replace $u(x,t) = N(x,t)w(x)$ by $\phi_n(x,t)$, its trigonometric interpolant at the $2n$ points $x_{n,k}$, and replace $w(x_{n,k})$ everywhere by the approximation $\widetilde{w}_{n,k}$. Finally, we set $t = x_{n,j}$, $j = 0, 1, \dots, 2n-1$, everywhere. This results in the following $2n$ equations in the $2n$ unknowns $\widetilde{w}_{n,k}$:

$$\lambda\widetilde{w}_{n,j} + \frac{1}{2n}\sum_{k=0}^{2n-1}\left[{\sum_{q=-n}^{n}}'' M_{\sigma,q}e_q(x_{n,j} - x_{n,k})\right]N(x_{n,k}, x_{n,j})\widetilde{w}_{n,k} = a(x_{n,j}), \quad 0 \le j \le 2n-1. \tag{7.5}$$

Here we have invoked (4.8). Since the underlying numerical quadrature formula $Q_{\sigma,n}(t; N(\cdot,t)w(\cdot))$ has high accuracy, we expect the $\widetilde{w}_{n,k}$ to approximate the $w(x_{n,k})$ with high accuracy too.

### Data availability

No data was used for the research described in the article.

## Appendix. $H_\sigma(t; u)$ Via Richardson extrapolation

Let us assume that $u \in C^\infty(\mathbb{R})$ and let us express $H_\sigma(t; u)$, $\sigma \notin \mathbb{Z}$, in the form

$$H_\sigma(t; u) = I[f] = \,\,=\!\!\!\!\!\!\!\int_0^T f(x)\,dx, \quad f(x) = g(x)|x-t|^\sigma, \quad 0 < x, t < T, \tag{A.1}$$

where $g(x)$ is defined via $g(x)|x-t|^\sigma = \left|\sin\frac{\pi(x-t)}{T}\right|^\sigma u(x)$, thus

$$g(x) = \left(\frac{\pi}{T}\right)^\sigma \left|\operatorname{sinc}\left(\frac{\pi(x-t)}{T}\right)\right|^\sigma u(x). \tag{A.2}$$

Note that we have written $f(x)$ and $g(x)$ and not $f(x,t)$ and $g(x,t)$ since $t \in (0,T)$ is being held fixed. Of course, neither $g(x)$ nor $|x-t|^\sigma$ is $T$-periodic even though $f(x)$ is; nevertheless, $g \in C^\infty[0,T]$. Now, by $T$-periodicity of $f(x)$, we can express (A.1) also as

$$H_\sigma(t; u) = I[f] = \,\,=\!\!\!\!\!\!\!\int_t^{t+T} f(x)\,dx \tag{A.3}$$

and that $f(x)$ is singular at the new endpoints $x = t$ and $x = t+T$ and is infinitely differentiable on $(t, t+T)$, with asymptotic expansions

$$f(x) \sim \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{k!}(x-t)^{\sigma+k} \quad \text{as } x \to t+, \tag{A.4}$$

$$f(x) \sim \sum_{k=0}^{\infty} (-1)^k \frac{g^{(k)}(t)}{k!}(t+T-x)^{\sigma+k} \quad \text{as } x \to (t+T)-. \tag{A.5}$$

Let us now define the trapezoidal sum $\check{T}(h)$ for the integral $=\!\!\!\!\!\!\int_t^{t+T} f(x)\,dx$ as

$$\check{T}(h) = h\sum_{j=1}^{m-1} f(t+jh), \quad h = \frac{T}{m}, \quad m = 1, 2, \dots. \tag{A.6}$$

Clearly, $\check{T}(h)$ is well-defined since $f(x)$ is infinitely differentiable on $(t, t+T)$.

Applying Theorem 2.1 of Sidi [14] to $\check{T}(h)$ in (A.6), along with (A.3) and the asymptotic expansions of $f(x)$ in (A.4)–(A.5), recalling also that $\sigma \notin \mathbb{Z}$, we have the generalized Euler–Maclaurin expansion

$$\check{T}(h) \sim \,\,=\!\!\!\!\!\!\!\int_t^{t+T} f(x)\,dx + 2\sum_{k=0}^{\infty} \frac{g^{(2k)}(t)}{(2k)!}\zeta(-\sigma-2k)h^{\sigma+2k+1} \quad \text{as } h \to 0 \text{ (or } m \to \infty), \tag{A.7}$$

where $\zeta(z)$ is the Riemann Zeta function. Of course, when $\sigma > -1$, $=\!\!\!\!\!\!\int_t^{t+T} f(x)\,dx$ is a regular integral and (A.7) remains valid.

We now choose positive integers $1 \le m_0 < m_1 < m_2, \dots$, and set $h_i = T/m_i$, and apply the Richardson extrapolation process to the sequence $\{\check{T}(h_i)\}_{i=0}^{\infty}$ to approximate $I[f] = \,=\!\!\!\!\!\!\int_t^{t+T} f(x)\,dx$; see Sidi [17, Chapter 1]. We define our approximations $A_n^{(j)}$ to $I[f]$ via the linear equations

$$\check{T}(h_i) = A_n^{(j)} + \sum_{k=0}^{n-1} \alpha_k h_i^{\sigma+2k+1}, \quad i = j, j+1, \dots, j+n, \tag{A.8}$$

assuming that we do not want to bother with $g(x)$ and its derivatives. Here $\alpha_k$ are additional unknowns that are of no interest.

The sequences of approximations $\{A_n^{(j)}\}_{n=0}^{\infty}$ (with fixed $j$, such as $j = 0$) have the best convergence properties. For example, when the integers $m_i$ are chosen as $m_i = 2^i$, $i = 0, 1, \ldots$, the sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ converge to $\int_0^T f(x)\,dx$ spectrally, as shown in Sidi [17, Theorem 1.5.4]. [In this case, the cost of determining $A_n^{(j)}$ involves $2^{j+n}$ evaluations of $f(x)$.]

The $A_n^{(j)}$ can be determined, without having to solve the linear equations in (A.8), via the author's (recursive) W-algorithm [18] (see also [17, Section 7.2]) as follows:

1. For $j = 0, 1, \ldots$, set

$$M_0^{(j)} = \frac{\check{T}(h_j)}{h_j^{\sigma+1}}, \quad N_0^{(j)} = \frac{1}{h_j^{\sigma+1}}.$$

2. For $j = 0, 1, \ldots$, and $n = 1, 2, \ldots$, compute

$$M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{h_{j+n}^2 - h_j^2}, \quad N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{h_{j+n}^2 - h_j^2}.$$

3. For $j = 0, 1, \ldots$, and $n = 0, 1, \ldots$, compute

$$A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}}.$$

Note that we need to compute only the trapezoidal sums $\check{T}(h_i)$, $j \leq i \leq j+n$. We do not need to compute $g(x)$ and its derivatives. Mere knowledge of the powers $h^{\sigma+2k+1}$, $k = 0, 1, \ldots$, in the generalized Euler–Maclaurin expansion (A.7) is sufficient.

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